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A Note on the Eigenvalue Problem in Crystal Statistics.

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§ 1. Introduction.

In recent years there have been some remarkable developments in the mathematical theory of order-disorder transition. The problem of determining the partition function for a lattice with simple interaction model was found, by several authors,[1] to be reduced to an eigenvalue problem which is easier to handle with, and which, with suitable approximations, would reveal more refined details of the phase transition than the former theories. This method of attack culminated in the work of Onsager,[2] who succeeded in giving the rigorous solution of the above mentioned eigenvalue problem for two-dimensional simple square Ising lattice. The mathematical tools involved, however, are so hopelessly complicated that one would quite simply lose sight in the jungle of hypercomplex numbers. A considerable improvement on this mathematical point has been attained by Husimi and Syôzi[3] who used it to solve the eigenvalue problem for the honeycomb type lattice. The present author, independently of them, has reached more or less similar ideas and considerations which it is the purpose of the present paper to expose in brief detail. Though as yet no substantial applications has been attempted, nor anything physically new has been derived, it may be hoped that it will do some profit for those who are interested in such problems.

§ 2. Preliminary Considerations.

Let us take N identical particles 1, 2,$\cdots N$, arranged in order, and consider the interchange $P_{n,n+1}$ of two adjacent particles n and $n+1$ ($n=1, 2,\cdots N$). Such operators are met with in the theory of ferromagnetism in which the eigenvalues of the operator (for one-dimensional case)

$$P=\sum_n P_{n,n+1}=\sum \frac{1+\sigma_n\cdot\sigma_{n+1}}{2} \tag{1}$$

are asked. Here σ_n means the spin matrix $(\sigma_x, \sigma_y, \sigma_z)$ at the nth site. As P commutes with $\sum_n \sigma_{nz}=2S-N$, where S is the total number of spins oriented to $+z$ direction, we can work within a subspace in which S is kept fixed. Then

P is equivalent to an operator affecting the arrangement of S particles over N sites. Each site is either occupied or not occupied by one of these particles. If we introduce such operators, that change the unoccupied nth site into occupied, and *vice versa*, by

$$a_n^+=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad a_n=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{2}$$

then P can be written in terms of them as follows:

$$\begin{aligned} P&=\sum(a_n^+a_{n+1}+a_{n+1}^+a_n+a_n^+a_na_{n+1}^+a_{n+1}+a_na_n^+a_{n+1}a_{n+1}^+) \\ &=\sum(a_n^+a_{n+1}+a_{n+1}^+a_n-2a_n^+a_n+2a_n^+a_na_{n+1}^+a_{n+1})+N. \end{aligned} \tag{3}$$

The operators (2) remind one of the operators familiar in the second quantization, the only difference being the lack of anticommutativity between different sites. But we can easily see that the introduction of the sign functions necessary to make the a_n's and a_n^+'s anticommute does not impair the relation (3), so that we are permitted to put

$$[a_n,\ a_m^+]_+=\delta_{nm}, \quad [a_n,\ a_m]_+=[a_n^+,\ a_m^+]_+=0. \tag{4}$$

Now we make use of the Fourier transformation to rewrite (3) as

$$\begin{aligned} P&=\sum_{n,m}(a_n^+a_m\delta(n-m+1)-a_n^+a_m\delta(n-m)+a_n^+a_na_m^+a_m\delta(n-m+1)+\text{conj.} \\ &=\frac{1}{N}\sum_{n,m}\sum_k\left(a_n^+e^{-\frac{2\pi i}{N}nk}a_me^{+\frac{2\pi i}{N}mk}e^{-\frac{2\pi i}{N}k}-a_n^+e^{-\frac{2\pi i}{N}nk}a_me^{+\frac{2\pi i}{N}mk}+\text{conj.}\right) \\ &\qquad +\sum_{n\,m}a_n^+a_na_m^+a_m\delta(n-m\pm1) \\ &=\sum_k\left(a_k^+a_ke^{-\frac{2\pi i}{N}k}-a_k^+a_k+\text{conj.}\right)+\sum a_n^+a_na_m^+a_m\delta(n-m\pm1), \end{aligned} \tag{5}$$

where

$$a_k\equiv\frac{1}{\sqrt{N}}\sum_n a_ne^{\frac{2\pi i}{N}\cdot nk}, \quad a_k^+\equiv\frac{1}{\sqrt{N}}\sum_n a_ne^{-\frac{2\pi i}{N}nk}. \tag{5'}$$

There is a striking analogy to the ordinary quantum mechanics. P is just the Hamiltonian of interacting identical systems for which the "kinetic energy" and the "interaction potential" are given respectively by

$$2\left(\cos\frac{2\pi}{N}k-1\right) \quad \text{and} \quad \delta(x_i-x_j\pm1), \tag{6}$$

where k and x take discrete values. The kinetic terms alone lead to the well known Bloch spin waves whereas the interaction terms cause scattering of them. The latter can be neglected when the number of particles are small because the interaction force is of short range, which corresponds to the fact that the notion of spin wave is a good approximation when the magnetization is nearly complete (low temperature).

As the above example shows, the introduction of a set of anticommuting operators is particularly convenient because of its orthogonal property[4]. Instead of the creation and annihilation operators a_n^+ and a_n, we can also use the quantities $a_n^+ + a_n$ and $i(a_n^+ - a_n)$, or in other words, a set of anticommuting quantities with the commutation relations

$$[x_r,\ x_s]_+ = 2\delta_{rs}, \quad r,\ s = 1, \cdots 2N. \tag{7}$$

According to the theory of spinors, such quantities can be considered as making up an orthogonal basis for a $2N$–dimentional vectors space[5]. A product of two vectors $x_r x_s$ commutes with x_t if $t \neq r,\ s$. Thus we have a means for expressing commutable quantities in terms of anticommutable quantities.

As a next example we shall take the eigenvalue problem for the operator

$$P' = \sum_{n=1}^{N} (\sigma_{nx}\sigma_{n+1,x} + \sigma_{ny}\sigma_{n+1,y}) \equiv \sum (A_n + B_n), \tag{8}$$

which is a part of the operator P defined in (1). The commutation relations for A and B are

$$\begin{aligned} &[A_n,\ A_m]_- = [B_n,\ B_m]_- = 0, \\ &[A_n,\ B_m]_- = 0 \quad \text{if} \quad n = m \quad \text{or} \quad |\, n - m \,| \neq 1, \\ &[A_n,\ B_{n\pm1}]_+ = 0, \quad A_n^2 = B_n^2 = 1. \end{aligned} \tag{9}$$

That is, A_n and B_n anticommute when they are adjacent, otherwise simply commuting. Such quantities can equally be composed of totally anticommuting operators. Indeed, let us introduce $2N$ basic vectors $x_1,\ x_2, \cdots x_{2N}$ with the commutation relations (7), and define (Fig. 1), for N=even,

$$\begin{aligned} &A_n = i x_{2n} x_{2n+1}, \\ &\qquad n = 1,\ 2, \cdots N;\ 2N+1 \equiv 1. \\ &B_n = i x_{2n-1} x_{2n+2}. \end{aligned} \tag{10}$$

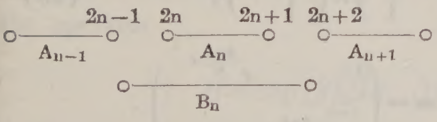

Fig. 1.

Obviously these quantities satisfy all the relations. (8). The only difference is as follows. From (8) we have

$$\prod_{n=1}^{N} A_n = \prod_{n=1}^{N} B_n = 1, \tag{11}$$

while from (10), on the other hand,

$$\begin{aligned} \Pi A_n &= i^N x_2 x_3 \cdots x_{2N} x_1 = -i^N x_1 x_2 \cdots x_{2N} \equiv x = \pm 1, \\ \Pi B_n &= i^N x_1 x_4 \cdot x_3 x_6 \cdot x_5 x_8 \cdots x_{2N-1} x_2 \\ &= i^N (-1)^{N-1} x_1 x_2 \cdots x_{2N} = (-1)^N x = x, \end{aligned} \tag{11'}$$

so that the operators in (8) cover only a subspace of the operator domains for (10). We have deliberately to select those solutions which lie in the subspace $x=1$.

Now (8) can be written as

$$P'=i\sum_{n=1}^{N}(x_{2n}x_{2n+1}+x_{2n-1}x_{2n+2})=i\sum_{n=1}^{N}(x_{2n}x_{2m+1}+x_{2n-1}x_{2m+2})\delta_{nm}$$

$$=\frac{i}{N}\sum_{k=1}^{N}\sum_{n,m}(x_{2n}x_{2m+1}+x_{2n-1}x_{2m+2})e^{\frac{2\pi k}{N}i(n-m)}=2\sum_{k=1}^{N}(x_ky_{-k}+y_kx_{-k}e^{\frac{2\pi k}{N}2i})$$

$$=2i\sum_{k=1}^{N/2}\{x_ky_{-k}(1-e^{-\frac{2\pi k}{N}2i})+x_{-k}y_k(1-e^{\frac{2\pi k}{N}2i})\}$$

$$=-4\sum_{k=1}^{N/2}(x_ky_{-k}e^{-\frac{2\pi k}{N}i}-x_{-k}y_ke^{\frac{2\pi k}{N}i})\sin\frac{2\pi k}{N}\equiv-4\sum_{k=1}^{N/2}z_k\sin\frac{2\pi k}{N}, \tag{12}$$

where

$$x_k\equiv\frac{1}{\sqrt{2N}}\sum_{n=1}^{N}x_{2n}e^{\frac{2\pi k}{N}ni},\quad y_k\equiv\frac{1}{\sqrt{2N}}\sum_{n=1}^{N}x_{2n+1}e^{\frac{2\pi k}{N}ni},$$

$$[x_k,\ x_{-l}]_+=[y_k,\ y_{-l}]_+=\delta_{kl},\quad k,\ l=-N,\ -(N-1),\cdots+N, \tag{12'}$$

$$[z_k,\ z_l]_-=0,\quad [x_k,\ y_{-l}]_+=0.$$

Thus we see that the eigenvalue of P' is a sum of the contributions from individual terms with z_k, which have four eigenvalues 0, 0, ±1 as shown by matrix representation:

$$x_k=\begin{pmatrix} & 1\\ 0 & \end{pmatrix}\times\begin{pmatrix}1 & \\ & -1\end{pmatrix},\quad x_{-k}=\begin{pmatrix} & 0\\ 1 & \end{pmatrix}\times\begin{pmatrix}1 & \\ & -1\end{pmatrix},$$

$$y_k=\begin{pmatrix}1 & \\ & 1\end{pmatrix}\times\begin{pmatrix} & 1\\ 0 & \end{pmatrix},\quad y_{-k}=\begin{pmatrix}1 & \\ & 1\end{pmatrix}\times\begin{pmatrix} & 0\\ 1 & \end{pmatrix}, \tag{13}$$

$$\therefore\quad z_k=x_ky_{-k}e^{-\frac{2\pi k}{N}i}-x_{-k}y_ke^{\frac{2\pi k}{N}i}=-\begin{pmatrix}0 & e^{\frac{2\pi k}{N}i}\\ e^{-\frac{2\pi k}{N}i} & 0\end{pmatrix}.$$

Then

$$P'=4\sum_{k=1}^{N/2}\varepsilon_k\sin\frac{2\pi k}{N},\quad \varepsilon_k=0,\ 0,\ 1\ \text{ or }\ -1. \tag{14}$$

The restriction $x=1$ becomes as follows. We write

$$\prod_{n=1}^{N}(\cos\theta+\sin\theta x_{2n}x_{2n+1})=\prod\exp(\theta x_{2n}x_{2n+1})=\exp[\theta\sum x_{2n}x_{2n+1}] \tag{15}$$

$$=\exp[2\theta\sum_{k=1}^{N/2}(x_ky_{-k}+x_{-k}y_k)]\equiv\prod_{k=1}^{N/2}[2\theta iR_k],$$

$$=\Pi\{1+R_k^2(\cos 2\theta-1)+i\sin 2\theta R_k\}, \tag{15'}$$

since
$$R_k^3=R_k. \tag{16}$$

Putting $\theta=\pi/2$, we have

$$(-1)^{\frac{N}{2}}x=\prod_{k=1}^{N/2}(1-2R_k^2). \tag{17}$$

R_k turns out to commute with z_k and has the eigenvalues ε_k^2. Hence we get the condition

$$(-1)^{N/2}x=\Pi(1-2\varepsilon_k^2)=(-1)^{N/2}. \tag{18}$$

which determines the number of non-zero ε_k.

Thus we see that the use of anticommuting quantities is successful for the operator (8). The problem of diagonalizing the operator has been reduced to diagonalizing a quadratic form. However, this ceases to be the case when we take instead of (8) the operator P defined in (1), which calls for a quartic form rather than a quadratic.

§ 3. Onsager's Problem.

Now let us take up the case of Onsager[2], that is, a plane square lattice with interaction according to Ising model:

$$E=J\sum s_i s_i'+J'\sum s_i s_{i+1}, \tag{19}$$

where s_i represents the spin $(=\pm1)$ of the ith site on a certain row and s_i' that of the neighboring one on the next row. The summation extends over all the sites in the lattice. According to Onsager, the problem of obtaining the partition function $F=Tr\exp(-E/kT)$ is reduced to asking for the largest eigenvalue of the operator

$$H=\exp[\beta\sum_{n=1}^{N}s_n s_{n+1}]\exp[\alpha\sum_{n=1}^{N}c_n],\quad N+1\equiv 1, \tag{20}$$

where
$$s_n^2=c_n^2=1,$$
$$[s_n,c_n]_+=0, \tag{20'}$$
$$[s_u,s_m]_-=[s_n,c_m]_-=[c_n,c_m]_-=0,\quad n\neq m.$$

We shall not occupy ourselves with the deduction of the formula (20), but suppose it as given, and try to solve the eigenvalue problem for H. Writing (20) simply as

$$\begin{aligned}&H=\exp[\beta\sum S_n]\exp[\alpha\sum C_n],\\&S_n\equiv s_n s_{n+1},\quad C_n\equiv c_n,\end{aligned} \tag{21}$$

we see that S_n and C_n have the properties

$$\begin{aligned} &[S_n, C_{n\pm 1}]_+ = 0, \quad \text{otherwise all commuting, and} \\ &S \equiv S_1 S_2 \cdots S_N = 1, \quad C \equiv C_1 C_2 \cdots C_N = \pm 1. \end{aligned} \tag{21'}$$

In view of the relations (20) and (21′), a natural idea which suggests itself is to express H in terms of the operators of the second quantization as was illustrated in the last section. Indeed, we can put

$$\begin{aligned} &s_n s_{n+1} = i(a_n^+ - a_n)(a_{n+1}^+ + a_{n+1}), \quad c_n = 2a_n^+ a_n - 1, \\ &a_n^+ + a_n = c_1 c_2 \cdots c_{n-1} s_n, \quad i(a_n^+ - a_n) = c_1 c_2 \cdots c_n s_n, \end{aligned} \tag{22}$$

with

$$[a_n, a_m^+]_+ = \delta_{nm}.$$

Here c_n plays just the rôle of the sign function. Such a procedure has been independently devised by Husimi. Instead of (22) we may directly start from the anticommuting orthogonal basis $\{x_n\}$. Thus we replace S_n and C_n by

$$\begin{aligned} &S_n = i x_{2n} x_{2n+1}, \quad C_n = i x_{1n-1} x_{2n}, \\ &[x_r, x_s]_+ = 2\delta_{rs}, \quad r, s = 1, 2, \cdots 2N, \\ &S = \Pi S_n = i^N x_2 x_3 \cdots x_{2N} x_1 = -X,. \\ &C = \Pi C_n = i^n x_1 x_2 \cdots x_{2N} \equiv X. \end{aligned} \tag{23}$$

This substitution corresponds to the case $C=-1$ if we confine ourselves to the subspace $X=1$. The case $C=1$ obtains when one of the defining equations, say for C_1, is substituted by

$$C_1 = -i x_1 x_2. \tag{23'}$$

However, such minor modifications do not affect the final result for large N except some delicate problems like the boundary tension, so that we shall look apart from the eigenvalues of S and C in order to avoid unnecessary complexity. With the choice (20), the operator H can now be written as

$$H = \exp[i\beta \sum x_{2n} x_{2n+1}] \exp[i\alpha \sum x_{2n-1} x_{2n}] \equiv H_2 H_1. \tag{24}$$

As mentioned before, the quantities $x_1, x_2, \cdots x_{2N}$ may be regarded as orthogonal basic vectors in a $2N$–dimensional space. In this space, a rotation of angle θ in the plane spanned by x_r and x_s is expressed by the operator

$$\begin{aligned} &x_n \to U^{-1} x_n U, \\ &U = \exp[(\theta/2) x_r x_s] = \cos(\theta/2) + \sin(\theta/2) x_r x_s. \end{aligned} \tag{25}$$

In fact it is easily verified that

$$\begin{aligned} &U^{-1} x_r U = \cos\theta x_r + \sin\theta x_s, \\ &U^{-1} x_s U = -\sin\theta x_r + \cos\theta x, \\ &U^{-1} x_n U = x_n, \quad n \neq r, s. \end{aligned} \tag{26}$$

Then (24) means nothing but a product of rotations, first by angle $2\alpha i$, in the planes $x_{2n-1}x_{2n}$, and then by angle $2\beta i$, in the planes $x_{2n}x_{2n+1}$. Let us suppose that in a suitable coordinate system $\{x_n'\}$ the rotation H is brought to Jordan's standard form, so that

$$H=\exp\left[i\sum x_{2n}'x_{2n+1}'\gamma_n\right]. \tag{27}$$

Since $|ix_n'x_{n+1}'|=1$, the eigenvalue of the operator H that operates on a eigenvector Ψ is found in the expression

$$H\Psi=\exp\left[\sum\varepsilon_n\gamma_n\right]\Psi, \quad \varepsilon_n=\pm1. \tag{28}$$

We have only to solve the problem in the $2N$–dimensional subspace $\{x_n\}$ of basic vectors, instead of the whole space of Ψ with 2^N dimensions. By the first rotation H_1, a vector (x_n) is transformed into another vector (y_n), which, by the second rotation H_2, again changes into a third (z_n). Thus

$$\begin{aligned} y_{2n-1}&=\cos 2\alpha' x_{2n-1}+\sin 2\beta' x_{2n}, \\ y_{2n}&=-\sin 2\alpha' x_{2n-1}+\cos 2\beta' x_{2n}, \end{aligned} \tag{28}$$

$$\begin{aligned} z_{2n}&=\cos 2\beta' y_{2n}+\sin 2\beta' y_{2n+1}, \\ z_{2n+1}&=-\sin 2\beta' y_{2n}+\cos 2\beta' y_{2n+1}, \end{aligned} \tag{28'}$$

or

$$\begin{aligned} z_{2n}&=-bx_{2n-1}+dx_{2n}+ax_{2n+1}+cx_{2n+2}; \\ z&=-ax_{2n}+dx_{2n+1}+bx_{2n+2}+cx_{2n-1}. \end{aligned} \tag{28''}$$

Here $\alpha'=\alpha i$, $\beta'=\beta i$,

$$\begin{aligned} a&=\cos 2\alpha' \sin 2\beta', \quad b=\sin 2\alpha' \cos 2\beta', \\ c&=\sin 2\alpha' \sin 2\beta', \quad d=\cos 2\alpha' \cos 2\beta'. \end{aligned} \tag{28'''}$$

In matrix form,

$$H=\left(\begin{array}{cccccc} & \ddots & & & & \\ -b & d & a & c & & \\ & & \ddots & & & \\ c & -a & d & b & & \\ & & & \ddots & & \\ & & -b & d & a & c \\ & & & & \ddots & \\ & & c & -a & d & b \\ & & & & & \ddots \end{array}\right)\begin{array}{l} \\ \\ \\ \cdots 2n \\ \\ \cdots 2n+1 \\ \\ \\ \\ \end{array} \tag{29}$$

The eigenvalue equation for H becomes

$$\begin{aligned} -bx_{2n-1}+(d-\lambda)x_{2n}+ax_{2n+1}+cx_{2n+2}&=0, \\ cx_{2n-1}-ax_{2n}+(d-\lambda)x_{2n+1}+bx_{2n+2}&=0, \end{aligned} \tag{30}$$

or

$$\begin{aligned} ax_{2n+1}+cx_{2n+2}&=bx_{2n-1}+(\lambda-d)x_{2n}, \\ (\lambda-d)x_{2n+1}-bx_{2n+2}&=cx_{2n-1}-ax_{2n}. \end{aligned} \tag{30'}$$

Let us write

$$(x_{2n+1},\ x_{2n+2})=\psi_{n+1}\,,\quad (x_{2n-1},\ x_{2n})=\psi_n\,, \tag{31}$$

and

$$A=\begin{pmatrix} a, & c \\ \lambda-d, & -b \end{pmatrix},\quad B=\begin{pmatrix} b, & \lambda-d \\ c, & -a \end{pmatrix}. \tag{31'}$$

Then (30′) means

$$A\psi_{n+1}=B\psi_n\,,\quad \text{or}\quad \psi_{u+1}=A^{-1}B\psi_n\equiv C\psi_n\,, \tag{32}$$

hence $$\psi_n=C^{n-1}\psi_1\,. \tag{32'}$$

The periodicity condition requires that $\psi_{N+1}=\psi_1$, or

$$(E-C^N)\psi_1=0,\quad \det|E-C^N|=0. \tag{33}$$

If we denote the Nth roots of unity by $\xi_k=\exp(2\pi ki/N)=\exp(\varphi_k i)$, $k=1, 2, \cdots N$, then

$$\det|\xi_k E-C|=0,\quad \text{or}\quad |\xi_k A-B|=0, \tag{33'}$$

that is,

$$\begin{vmatrix} \xi_k a-b, & \xi_k c-(\lambda-d) \\ \xi_k(\lambda-d)-c, & -\xi_k b+a \end{vmatrix}=0. \tag{33''}$$

This gives a quadratic equation for λ:

$$\xi\lambda^2-\lambda[2\xi d+\xi^2 c+c]+\xi(a^2+b^2+c^2+d^2)-(1+\xi^2)(ab-cd)=0. \tag{34}$$

Substituting (28‴),

$$a^2+b^2+c^2+d^2=1,\quad ab-cd=0,$$

$$\lambda^2-\lambda[2d+(\xi+\xi^{-1})c]+1=0,$$

or $$\lambda^2-2\lambda(\cos 2\alpha'\cos 2\beta'+\sin 2\alpha'\sin 2\beta'\cos\varphi_k)+1=0. \tag{34'}$$

Putting the two roots as $\exp(\pm 2\gamma_k' i)$,

$$(1/2)(e^{2\gamma' i}+e^{-2\gamma' i})=\cos 2\gamma_k'=\cos 2\alpha'\cos 2\beta'+\sin 2\alpha'\sin 2\beta'\cos\varphi_k\,. \tag{35}$$

Returning to the original constants α and β, and writing $\gamma=\gamma' i$, we get just the relation given by Onsager:

$$\cosh 2\gamma_k=\cosh 2\alpha\cosh 2\beta-\sinh 2\alpha\sinh 2\beta\cos\varphi_k,\quad \varphi_k=2\pi k/N. \tag{35'}$$

The largest eigenvalue of H becomes then, according to (28),

$$H_{\max}=\exp[\Sigma|\gamma_k|]\,. \tag{36}$$

Such a procedure also applies, *mutatis mutandis*, to certain variants of Onsager model, e.g. the honeycomb lattice treated by Husimi and Syôzi. It fails, however, in case of those lattices with too many nearest neighbours, in particular the three-dimensional lattices. For we cannot express a quantity which anticommutes with more than two neighbors by a product of two totally anticommuting vectors. In fact the three-dimensional model necessitates us to resort to products of four x's as illustrated below:

$$
\begin{aligned}
H &= \exp[\beta\sum(s_{nm}s_{n,m+1}+s_{nm}s_{n+1,m})\exp[u\sum c_{nm}] \\
&= \exp[i\beta\sum h_r]\exp[u\sum k_s], \qquad (37) \\
h &= x_1x_3',\ x_4x_2'',\quad k=x_1x_2x_3x_4. \quad \text{(See Fig. 2.)}
\end{aligned}
$$

Supplementary conditions are also to be considered. The occurence of quartic forms may look reasonable when we note the so-called dual transformation. Such transformations do really exist even in the three-dimensional case. In fact let us take a cubic lattice with atoms located at the middle point of each edge of the cell, and let each set of the four atoms lying on a side plane determine a four-body interaction energy $\mu s_1s_2s_3s_4$, $s_i=\pm 1$. Then it is easy to show that this lattice is dual to a simple cubic lattice made up of the body center sites of the former with ordinary two-body interactions. Perhaps the four-body force may have some bearing on the quantity k_s in (37).

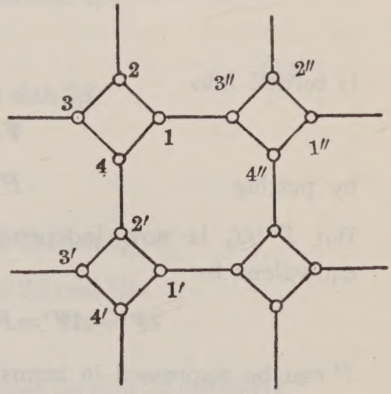

Fig. 2.

§ 4. Kramers-Wannier's method of approach.

Our next investigations concern the screw lattice model adopted by Kramers and Wannier for solving the two-dimensional case. This model seems more convenient for general purposes than that used by Onsager. Now let us try to transform the Onsager model into the K-W model as follows. Rewrite (20) (after reversing the order of H_1 and H_2) as

$$
\begin{aligned}
H_1H_2 &= \exp[i\alpha x_1x_2]\exp[i\alpha x_3x_4]\cdots\exp[i\beta x_2x_3].\exp[i\beta x_4x_5]\cdots \\
&= \exp[i\alpha x_1x_2]\exp[i\alpha x_3x_4]\exp[i\beta x_2x_3]\exp[i\alpha x_5x_6]\exp[i\beta x_4x_5]\cdots \\
&= \exp[i\alpha x_1x_2]\prod_{n=1}^{N-1}(\exp[iux_{2n+1}x_{2n+2}]\exp[i\beta x_{2u}x_{2n+1}]\exp[i\beta x_{2N}x_1], \qquad (38)
\end{aligned}
$$

which can be modified, without grave consequences, to

$$
\begin{aligned}
H &= \prod_{n=1}^{N}\exp[iux_{2n+1}x_{2n+2}]\exp[i\beta x_{2n}x_{2n+1}] \equiv \Pi H_n, \\
H_n &\equiv \exp[iux_{2n+1}x_{2n+2}]\exp[i\beta x_{2n}x_{2n+1}]. \qquad (38')
\end{aligned}
$$

It turns out more convenient to consider an operator which has its component factors H_n in the reverse order to (51) (which is also equivalent to a renumbering of the order). Then define a displacement operator P, which changes x_n into x_{n+1}, and put

$$\Psi_n = H_n H_{n-1} \cdots H_1 \Psi_0 \tag{39}$$

with obvious relations

$$H_n = P^n H_0 P^{-n}, \quad H_0 = \exp[i\beta x_{2N} x_1] \exp[i\alpha x_1 x_2]\,. \tag{39'}$$

The relation (a Schrödinger equation for discrete time variable!)

$$\Psi_{n+1} = H_n \Psi_n \tag{40}$$

is turned into

$$\Psi_{n+1}' = P^{-1} H_0 \Psi_n' \equiv A \Psi_n' \tag{40'}$$

by putting

$$P^{-n} \Psi_n = \Psi_n' \quad \text{or} \quad \Psi_n = P^n \Psi_n'. \tag{40''}$$

But $P^{-1}H_0$ is now independent of n, hence the eigenvalue problem for Ψ_n is equivalent to

$$\lambda \Psi' = A \Psi' = P^{-1} H_0 \Psi', \quad \lambda^N = E, \quad (H\Psi_0 = \Psi_N = E\Psi_0). \tag{41}$$

P can be expressed in terms of the basic vectors as

$$\begin{aligned} P = {} & \exp\left[\frac{\pi}{4} x_1 x_{2N-1}\right] \exp\left[-\frac{\pi}{4} x_1 x_{2N-3}\right] \cdots \exp\left[\pm\frac{\pi}{4} x_1 x_3\right] \\ & \times \exp\left[\frac{\pi}{4} x_2 x_{2N}\right] \exp\left[-\frac{\pi}{4} x_2 x_{2N-2}\right] \cdots \exp\left[\pm\frac{\pi}{4} x_2 x_4\right] \end{aligned} \tag{42}$$

because of the relations

$$\begin{aligned} & \exp\left[\frac{\pi}{4} x_1 x_2\right] x_1 \exp\left[-\frac{\pi}{4} x_1 x_2\right] \\ & = \frac{1}{2}[1 - x_1 x_2] x_1 [1 + x_1 x_3] = \frac{1}{2}(x_1 - x_1 + 2x_2) = x_2\,, \\ & \exp\left[\frac{\pi}{4} x_1 x_2\right] x_2 \exp\left[-\frac{\pi}{4} x_1 x^2\right] = \frac{1}{2}(x_2 - x_2 - 2x_1) = -x_1\,. \end{aligned} \tag{42'}$$

Thus the operator A can again be regarded as a product of rotation operators. Let us look for an eigenvector of the form

$$x = \sum_{s=1}^{N-1} \lambda^{n-1} (a x_{2n-1} + b x_{2n}) + c x_{2N} + \lambda^{N-1} a x_{2N-1}\,. \tag{43}$$

Substituting in the equation $AxA^{-1} = \lambda x$, we get the following result:

$$\left.\begin{aligned}&\cos 2\alpha'\cos 2\beta' x-\sin 2\alpha'\cos 2\beta' y+\sin 2\beta' z=\lambda^N x\,,\\&\sin 2\alpha' x+\cos 2\alpha' y=\lambda z\,,\\&-\sin 2\beta'\cos 2\alpha' x+\sin 2\alpha'\sin 2\beta' y+\cos 2\beta' z=\lambda^{N-1}y\,,\end{aligned}\right\}\tag{44}$$

or

$$\left.\begin{aligned}&\cosh 2\alpha\cosh 2\beta x+i\sinh 2\alpha\cosh 2\beta y-i\sinh 2\beta z=\lambda^N x\,,\\&-i\sinh 2\alpha x+\cosh 2\alpha y=\lambda z\,,\\&i\sinh 2\beta\cosh 2\alpha x-\sinh 2\alpha\sinh 2\beta y+\cosh 2\beta z=\lambda^{N-1}y\,.\end{aligned}\right\}\tag{44'}$$

Eliminating z by virture of the second equation,

$$\begin{aligned}&(\cosh 2\alpha\cosh 2\beta-\sinh 2\alpha\sinh 2\beta\lambda^{-1})x+i(\sinh 2\alpha\cosh 2\beta\\&\qquad\qquad-\sinh 2\beta\cosh 2\alpha\lambda^{-1}y=\lambda^N x\,,\\&i(\sinh 2\beta\cosh 2\alpha-\sinh 2\alpha\cos 2\beta\lambda^{-1})x-(\sinh 2\alpha\sinh 2\beta\\&\qquad\qquad-\cosh 2\beta\cosh 2\alpha\lambda^{-1}y=\lambda^{N-1}y\,,\end{aligned}\tag{44''}$$

which determine λ according to

$$\begin{aligned}&\lambda^{2N-1}+\sinh 2\alpha\sinh 2\beta\lambda^N+(-\cosh 2\beta\cosh 2\alpha-\cosh 2\alpha\cosh 2\beta)\lambda^{N-1}\\&\quad+\sinh 2\alpha\sinh 2\beta\lambda^{N-2}+(\sinh^2 2\alpha\sinh^2 2\beta+\cosh^2 2\beta\cosh^2 2\alpha\\&\quad-\sinh^2 2\alpha\cosh^2 2\beta-\sinh^2 2\beta\cosh^2 2\alpha\lambda^{-1}\\&\quad+(-\sinh 2\alpha\cosh 2\alpha\sinh 2\beta\cosh 2\beta+\sinh 2\alpha\cosh 2\alpha\sinh 2\beta\cosh 2\beta)\lambda^{-2}\\&\quad-\cosh 2\alpha\sinh 2\alpha\cosh 2\beta\cosh 2\beta+\sinh 2\alpha\cosh 2\alpha\sinh 2\beta\cosh 2\beta=0.\end{aligned}$$

This in turn reduces to

$$\lambda^{2N}+\sinh 2\alpha\sinh 2\beta(\lambda+\lambda^{-1})-2\cosh 2\alpha\cosh 2\beta+\lambda^{-N}=0.\tag{45'}$$

Putting $\lambda=\exp 2\gamma$,

$$\cosh 2N\gamma=\cosh 2\alpha\cosh 2\beta-\sinh 2\alpha\sinh 2\beta\cosh\gamma.\tag{46}$$

This is the equation for γ. To solve this, we assume that $2\gamma=2\gamma_0+\omega i$, and substitute in (46), thereby making use of the relation

$$\cosh(2\gamma_0+\omega\iota)=\cosh 2\gamma_0\cos\omega+i\sinh 2\gamma_0\sin\omega\,.\tag{47}$$

Comparison of real and imaginary parts yields the relations

$$\begin{aligned}&\cosh 2N\gamma_0\cos N\omega=\cosh 2\alpha\cosh 2\beta-\sinh 2\alpha\sinh 2\beta\cosh\gamma_0\cos\omega\,,\\&\sinh 2N\gamma_0\sin N\omega=\sinh 2\alpha\sinh 2\beta\sinh\gamma_0\sin\omega\,.\end{aligned}\tag{48}$$

Choose $\omega=k\pi/N$, $k=1, 2,\cdots 2N$, so that $\sin N\omega=0$, $\cos N\omega=(-1)^k$. In the limit $N\to\infty$ and $N\gamma_0\to\Gamma=$finite, we have $\gamma_0=0$, and both equations become satisfied by this choice of ω. Thus

$$\pm\cosh 2\Gamma=\cosh 2\alpha\cosh 2\beta-\sinh 2\alpha\sinh 2\beta\cos\omega\,,\tag{49}$$

with $$\omega=k\pi/N. \tag{49'}$$

But the minus sign is impossible, hence the allowed values for ω are

$$\omega=2k\pi/N, \quad k=1, 2, \cdots N. \tag{49''}$$

This completely agrees with the previous results. The case of the three-dimensional lattice can be treated analogously, but we shall not give here the details.

§ 5. Additional Remarks.

The mathematical tricks employed in the preceding sections seem to allow a rather natural explanation from a somewhat more general point of view. The essential point is that we consider, instead of an ordinary eigenvalue problem, a different one of the form

$$[H, X]_-=\lambda X \tag{50}$$

for the "eigenoperator" X[6]. Important consequences drawn from this equation are: (a) λ is the difference of two eigenvalues of $H: \lambda=E_n-E_m$; (b) X transforms certain eigenvector Ψ_m of H with E_m into another one Ψ_n with eigenvalue $E_n=E_m+\lambda$; and (c) Product of two eigenoperators X_2X_1 is again an eigenoperator, with eigenvalue $\lambda=\lambda_1+\lambda_2$, transforming Ψ_m into a third eigenvector Ψ_l. If we integrate the relation (50) according to

$$\frac{d}{ds}X=[H, X], \tag{51}$$

we arrive at the equation

$$e^{sH}Xe^{-sH}=e^{\lambda}X. \tag{51}$$

The equation (51) just corresponds to the problem of obtaining the eigenoperator $\{x_n'\}$ for the rotation H encountered in Section 3 and 4. When the operator H has a simple structure, a general eigenoperator X will be factorized into a product of prime eigenoperators:

$$X=X_1X_2\cdots X_k,$$

with
$$e^{\lambda}=e^{\lambda_1+\lambda_2+\cdots\lambda_k}. \tag{52}$$

The largest λ is the difference of the largest and smallest eigenvalues of sH in (52). If the eigenvalues E_i are symmetrically distributed around zero, $E_{\max}$ will itself be given by $E_{\max}=\lambda_{\max}\ 2$, a fact which casts a sidelight on the procedure used in the previous sections.

The main part of the present work had been completed nearly two years ago. It is through the kindness of Professor Husimi and Mr. Syôzi of Osaka University that the author enjoys the opportunity of publishing this note. He wishes to express on this occasion his cordial thanks to them and also to Professor Kubo of Tokyo University for helpful discussions.

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(4) An alternative way for solving (3) is to use commutative, rather than anticommutative, operators which corresponds to Bose statistics, together with some supplementary conditions. This leads to Bethe's rigorous solution for one-dimensional model (ZS. f. Phys. **71** (1931), 205; *Handbuch d. Phys.* XXIV/2, 604). We shall not here enter into details about it.

(5) Cf. Cartan: *Leçons sur la théorie des spineurs*, 1938.

(6) A deeper research on this subject will be found in: Y. Nambu, *On the method of the third quantization*, Prog. Theor. Phys. **4** (1949), 331 and 399.

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Five Dimensional Approach to Regularized Quantum Electrodynamics

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§ 1. Introduction

Recent researches developed by Tomonaga and his collaborators, Schwinger and Feynman have brought the theory of quantum electrodynamics to an advanced stage.. These authors have extended the field theories initiated by Heisenberg-Pauli's work fully relativistically, and have given them beautiful appearances. On the other hand, the development of ingenious and minute experiment which has brilliant representatives in the determination of energy levels of hydrogen atoms or of the value of the anomalous magnetic moment of an electron, makes it possible to estimate the effects of the interaction between the radiation field and charged particles. These situations force us to harmonize the frequently said divergence difficulties of quantum field theory with the real experimental facts. Bethe, Schwinger, Weisskopf, Tomonaga and others restored the idea of electromagnetic mass and constructed a formally closed theory by the use of "ronormalization" method and obtained satisfactory agreements with experiments though this method has an unsatisfactoriness of handling the quantities which are divergent in practice.

In spite of many successes, it was pointed out by many authors that these theories contain ambiguous features concerning with the photon self-energy problems and are being investigated in many other cases.

This defects apparently come from the fact that they are treating infinite quantities in separate way in each cases.

We believe that these divergence difficulties are closely connected with the infinitesimal character of the commutators of field variables and more essentially with the point character of the interaction between fields which was the corner stone of the present quantum field theories. Without altering this fundaments it would be impossible to get a satisfactory theory and the future theory would be the one in which no divergence quantities appear explicitly. Of course it may be probable that the present difficulties come from the defects of the mathematical devices. But in our opinion the improvement of such mathematical tools would necessarily cause the change of the present physical concepts.

Recently, Pauli and Villars[1] proposed a remedy for these difficulties. They have "regularized" the infinite quantities and removed ambiguities concerning them. Although this method contains some ambiguities in the case of γ-instability of neutral Bose particles[2)], we think that this is an good approximation to the correct theory in the sense that it gives satisfactory results.

It was shown that Pauli's method is equivalent to the mixed fields.[3)] We have brought these fields together in a closed form by extending the present theory to a five dimensional formulation, and got results in agreement with Pauli's one. In our formalism the added coordinate is independent from the usual space-time coordinates and thus has a conventional character. It is, so to speak, an "unphysical variable." We are reconciled ourselves, however, to this circumstance, though it is very unsatisfactory, in view of the transitional character of the present formalism. We think that the new freedoms, if it were necessary to add them, would not have the equivalent character to the space-time coordinates, and we hope that our formalism would become an approximation of some 5-dimensional covariant form.

§ 2. Free Field

i) C-number theory

We introduce formally the fifth coordinate x_5 which is supposed to have the space-like character. The distinction of the fifth freedom from others mainly appears in the transformation character of the field variables.

a) The electromagnetic field.

We assume the equation of motion*

$$\frac{\partial^2}{\partial x_k \partial x_k} U_i(x_1, x_2, x_3, x_4, x_5)=0 \qquad (i, k=1, 2, 3, 4, 5) \tag{2·1}$$

as the natural extension. (2·1) can be derived from the Lagrange function

$$L=\frac{1}{2}\frac{\partial U_i}{\partial x_k}\frac{\partial U_i}{\partial x_k}. \tag{2·2}$$

It is possible to construct the stress-tensor T_{ik} in analogous way to the usual case:

$$T_{ik}=\frac{1}{2}\left(\frac{\partial U_j}{\partial x_i}\frac{\partial U_j}{\partial x_k}+\frac{\partial U_j}{\partial x_k}\frac{\partial U_j}{\partial x_i}-\delta_{ik}\frac{\partial U_j}{\partial x_l}\frac{\partial U_j}{\partial x_l}\right). \tag{2·3}$$

It satisfies the conservation equation $\dfrac{\partial T_{ik}}{\partial x_i}=0.$

Here the physical meaning of U_5 comes into question. It may be possible to give it a property of mixing field (e.g. pseudo-scalar field). But we take U_5 as zero field throughout in the case of electromagnetic field, i.e.

$$U_5=0,$$

---

* Latin indices run from 1 to 5, while Greek indices from 1 to 4.

for the sake of simplicity.

If we develop U_μ $(\mu=1,2,3,4)$ as

$$U_\mu(x, x_5)=\int a(k_5)\, A_\mu(x, k_5)\, e^{ik_5x_5}dk_5, \tag{2·4}$$

where $a(k_5)$ is a some function of k_5 and has the following property

$$a(k_5)=a(-k_5),$$

then (2·1) is equivalent to the equation

$$\left(\frac{\partial^2}{\partial x_\lambda \partial x_\lambda}-k_5^2\right) A_\mu(x, k_5)=0. \tag{2·1'}$$

b) Electron field.

The equation of motion is

$$\left(\gamma_i\frac{\partial}{\partial x_i}+m\right)\psi(x_1, x_2, x_3, x_4, x_5)=0, \tag{2·5}$$

where $\gamma_i(i=1,2,3,4,5)$ is the usual γ-matrices and m is a constant representing intrinsic mass of the electron. And the Lagrange function is

$$L=-\frac{1}{2}\,\psi^+\left(\gamma_i\frac{\partial}{\partial x_i}+m\right)\psi, \tag{2·6}$$

where ψ^+ is the adjoint function to ψ.

Developing ψ^+, ψ as

$$\begin{aligned}\psi(x, x_5)&=\int b(k_5)\, e^{ik_5x_5}\,\phi(x, k_5)dk_5\,, \\ \psi^+(x, x_5)&=\int b^+(k_5)\, e^{-ik_5x_5}\,\phi^+(x, k_5)\, dk_5,\end{aligned} \tag{2·7}$$

(2.5) becomes

$$\begin{aligned}&\left(\gamma_\lambda\frac{\partial}{\partial x_\lambda}+m+i\gamma_5k_5\right)\phi(x, k_5)=0, \\ &\left(\tilde{\gamma}_\lambda\frac{\partial}{\partial x_\lambda}-m-i\gamma_5k_5\right)\phi^+(x, k_5)=0.\end{aligned} \tag{2·5'}$$

ii) Quantization

We assume the commutation relations of the field variables as

$$\begin{aligned}&[U_\mu(x_i),\ U_\nu(x_i')]=i\delta_{\mu\nu}\,\varepsilon(x_i-x_i'), \\ &\{\psi_\alpha(x_i),\ \psi_\beta^+(x_i')\}=\frac{1}{i}\left(\gamma_i\frac{\partial}{\partial x_i}-m\right)_{\alpha\beta} E(x_i-x_i'),\end{aligned} \tag{2·8}$$

the other commutators or anticommutators being zero. Here $\varepsilon(x_i)$, $E(x_i)$ must satisfy

$$\begin{aligned} &\frac{\partial^2}{\partial x_k \partial x_k}\varepsilon(x_i)=0, \\ &\left(\frac{\partial^2}{\partial x_k \partial x_k}-m^2\right)E(x_i)=0, \end{aligned} \tag{2·9}$$

and has the properties such that $\frac{\partial}{\partial x_0}\varepsilon(x_i)\Big|_{x_0=0}=\frac{\partial}{\partial x_0}E(x_i)\Big|=\delta(x_1)\delta(x_2)\delta(x_3)\int\rho(k_5)e^{ik_5k_5}dk_5$. In view of the usual form of (2·1′) and (2·5) we should give the commutators of A_μ or ϕ analogous forms to the usual ones. For this purpose we put

$$\begin{aligned} \varepsilon(x, x_5)&=\int dk_5\, \rho(k_5)\, e^{ik_5x_5}\, D_{k_5}(x), \\ E(x, x_5)&=\int dk_5\, \rho(k_5)\, e^{ik_5x_5}\, \Delta_{k_5}(x), \end{aligned} \tag{2·10}$$

where $\rho(k_5)=\rho(-k_5)$ is some function of k_5 and D_{k_5}, Δ_{k_5} are the usual invariant Δ-functions which satisfy the equation

$$\begin{aligned} &\left(\frac{\partial^2}{\partial x_\lambda \partial x_\lambda}-k_5^2\right)D_{k_5}(x)=0, \\ &\left(\frac{\partial^2}{\partial x_\lambda \partial x_\lambda}-m^2-k_5^2\right)\Delta_{k_5}(x)=0, \end{aligned} \tag{2·9′}$$

respectively. Then the commutation rules for A_μ, ϕ are

$$\begin{aligned} &[A_\mu(x, k_5), A_\nu(x', k_5')]=i\delta_{\mu\nu}\frac{\rho(k_5)}{|a(k_5)|^2}\,\delta(k_5+k_5')\,D_{k_5}(x-x'), \\ &\{\phi_\alpha(x, k_5), \phi_\beta^+(x', k_5')\}=\frac{1}{i}\,\frac{\rho(k_5)}{|b(k_5)|^2}\,\delta(k_5-k_5')\left(\gamma_\lambda\frac{\partial}{\partial x_\lambda}-m+i\gamma_5k_5\right)_{\alpha\beta}\Delta_{k_5}(x-x'). \end{aligned} \tag{2·8′}$$

However, if we want to decompose the field variables $U_\mu(x_i)$ and $\psi(x_i)$ into two parts, emission and absorption operators, some devices are taken as follows: that is, decomposing them

$$\begin{aligned} U_\mu(x_i)&=\overset{(+)}{U}_\mu(x_i)+\overset{(-)}{U}_\mu(x_i), \\ \psi(x_i)&=\overset{(+)}{\psi}(x_i)+\overset{(-)}{\psi}(x_i), \end{aligned} \tag{2·11}$$

we assume the commutation relations

$$\begin{aligned} &[\overset{(+)}{U}_\mu(x_i), \overset{(-)}{U}_\nu(x_i')]=i\delta_{\mu\nu}[\varepsilon(x_i-x_i')-i\varepsilon^{(1)}(x_i-x_i')], \\ &\{\overset{(+)}{\psi}_\alpha(x_i), \overset{(-)}{\psi}{}_\beta^+(x_i')\}=\frac{1}{i}\left(\gamma_i\frac{\partial}{\partial x_i}-m\right)_{\alpha\beta}[E(x_i-x_i')-iE^{(1)}(x_i-x_i')], \end{aligned} \tag{2·12}$$

where $\varepsilon^{(1)}$, $E^{(1)}$ have the following properties

$$\begin{aligned} &\frac{\partial^2}{\partial x_k \partial x_k}\,\varepsilon^{(1)}(x_i)=0, \\ &\left(\frac{\partial^2}{\partial x_k \partial x_k}-m^2\right) E^{(1)}(x_i)=0, \end{aligned} \tag{2·13}$$

which are the same form as (2·9). But we assume

$$\begin{aligned} \varepsilon^{(1)}(x_i)&=\int dk_5\, \sigma(k_5)\, e^{ik_5x_5}\, D^{(1)}_{k_5}(x), \\ E^{(1)}(x_i)&=\int dk_5\, \sigma(k_5)\, e^{ik_5x_5}\, \Delta^{(1)}_{k_5}(x), \end{aligned} \tag{2·14}$$

where $\sigma(k_5)=\sigma(-k_5)$ is not the same as $\rho(k_5)$. Then the equations (2·8′) decompose themselves into

$$\begin{aligned} &[\overset{(+)}{A}_\mu(x,k_5),\overset{(-)}{A}_\nu(x',k_5')]=i\delta_{\mu\nu}\delta(k_5+k_5')\,\frac{1}{|a(k_5)|^2}\,[\rho(k_5)D_{k_5}(x-x')-i\sigma(k_5)D^{(1)}_{k_5}(x-x')], \\ &\{\overset{(+)}{\phi}_\alpha(x,k_5),\overset{(-)}{\phi}_\beta(x',k_5')\}=\frac{1}{i}\,\delta(k_5-k_5')\frac{1}{|b(k_5)|^2}\left(\gamma_\lambda\frac{\partial}{\partial x_\lambda}-m+i\gamma_5k_5\right)_{\alpha\beta}\times \\ &\qquad\qquad \times[\rho(k_5)\Delta_{k_5}(x-x')-i\sigma(k_5)\Delta^{(1)}_{k_5}(x-x')]. \end{aligned} \tag{2·15}$$

To clarify the meaning of above expressions, we construct from the stress-tensor the energy $\overline{H}=\int T_{44}(dx_i)$ which can be written, for instance, for electromagnetic field in the following form

$$\overline{H}=\int\sqrt{\vec{k}^2+k_5^2}\left\{\frac{\rho(k_5)+\sigma(k_5)}{2}\,N_P(\vec{k},k_5)-\frac{\rho(k_5)-\sigma(k_5)}{2}\,N_N(\vec{k},k_5)\right\}d\vec{k}\,dk_5, \tag{2·16}$$

where

$$\begin{aligned} N_P(\vec{k},k_5)&=\frac{2\,|a(k_5)|^2}{\rho(k_5)+\sigma(k_5)}\,\overset{(-)}{A}_{P,\mu}(\vec{k},k_5)\overset{(+)}{A}_{P,\mu}(\vec{k},k_5), \\ N_N(\vec{k},k_5)&=\frac{2\,|a(k_5)|^2}{\rho(k_5)-\sigma(k_5)}\,\overset{(-)}{A}_{N,\mu}(\vec{k},k_5)\overset{(+)}{A}_{N,\mu}(\vec{k},k_5), \end{aligned} \tag{2·17}$$

and $\rho(0)=\sigma(0)=1$. $\overset{(-)}{A}_{\left\{\begin{smallmatrix}P\\N\end{smallmatrix}\right\}}(\vec{k},k_5)$ and $\overset{(+)}{A}_{\left\{\begin{smallmatrix}P\\N\end{smallmatrix}\right\}}(\vec{k},k_5)$ are the Fourier amplitudes of $A(x,k_5)$ and N_P, N_N have the values 0, 1, …… respectively. (See Appendix I.) Hence, $\rho(k_5)$ and $\sigma(k_5)$ mean the density of the distribution of particles with different mass state, and play the essential role for the later applications.

§ 3. Interaction between Electromagnetic Field and Electron. Self-Energy of the Electron.

When the electromagnetic field is present, the equation of motion of the electron becomes

$$\left[\gamma_\lambda\left(\frac{\partial}{\partial x_\lambda}+ieU_\lambda(x_i)\right)+m+\gamma_5\frac{\partial}{\partial x_5}\right]\psi(x_i)=0. \qquad (3\cdot1)$$

This means to add the Lagrange function a term

$$ieU_\mu(x_i)\,\psi^+(x_i)\,\gamma_\mu\,\psi(x_i). \qquad (3\cdot2)$$

Assuming the generalized Schrödinger equation

$$i\frac{\delta\Psi[\Sigma]}{\delta\Sigma(x, x_5)}=H(x, x_5)\,\Psi[\Sigma], \qquad (3\cdot3)$$

where $H=-j_\lambda(x, x_5)\,U_\lambda(x, x_5)$,

the calculation of various problems can be performed in entirely analogous way to the usual one. Σ means a space-like surface specified by the parameter x_i. Especially, we assume that the normal of this surface is always perpendicular to x_5-axis.

Starting from the state where an electron with "mass" m (ground mass state) is present, we can estimate the self-energy of the electron interacting with the electromagnetic field by the second order perturbation method. By the use of the notations analogous to that of Schwinger's paper[4)], we can write the self-energy operator of an electron with mass m as follows:*

$$\begin{aligned}H_{1(m),0}=-\frac{e^2}{8}\frac{1}{2\pi}\int dx_5'\int\{&[\psi^+(x,x_5),\gamma_\mu T^{(1)}(x-x',x_5-x_5')\gamma_\mu\psi(x',x_5')]_{1(m)}\bar{\varepsilon}(x-x',x_5-x_5')\\&+[\psi^+(x, x_5), \gamma_\mu\bar{T}(x-x', x_5-x_5')\gamma_\mu\psi(x', x_5')]_{1(m)}\,\varepsilon^{(1)}(x-x', x_5-x_5')\\&+[\psi^+(x', x_5')\gamma_\mu T^{(1)}(x'-x, x_5'-x_5)\gamma_\mu, \psi(x, x_5)]_{1(m)}\,\bar{\varepsilon}(x-x', x_5-x_5') \qquad (3\cdot4)\\&+[\psi^+(x', x_5')\gamma_\mu\bar{T}(x'-x, x_5'-x_5)\gamma_\mu, \psi(x, x_5)]_{1(m)}\,\varepsilon^{(1)}(x-x', x_5-x_5')\}dx',\end{aligned}$$

and in virtue of (2·17) this becomes

$$\begin{aligned}&-\frac{e^2}{8}\frac{1}{2\pi}\int dx_5'\int dx'\int dk_5\int dp_5\times\\&\times\frac{\rho(0)+\sigma(0)}{2}\{\rho(k_5)\sigma(p_5)\,e^{i(p_5+k_5)(x_5-x_5')}[\phi_p^+(x, 0), \gamma_\mu S_{p_5}^{(1)}(x-x')\gamma_\mu\phi_p(x', 0)]\bar{D}_{k_5}(x-x')\\&\quad+\rho(p_5)\sigma(k_5)\,e^{i(p_5+k_5)(x_5-x_5')}[\phi_p^+(x, 0), \gamma_\mu\bar{S}_{p_5}(x-x')\gamma_\mu\phi_p(x', 0)]\,D_{k_5}^{(1)}(x-x')\\&\quad+\rho(k_5)\sigma(p_5)\,e^{i(-p_5+k_5)(x_5-x_5')}[\phi_p^+(x', 0)\gamma_\mu S_{p_5}^{(1)}(x'-x)\gamma_\mu, \phi_p(x, 0)]\,\bar{D}_{k_5}(x-x')\\&\quad+\rho(p_5)\sigma(k_5)\,e^{i(-p_5+k_5)(x_5-x_5')}[\phi_p^+(x', 0)\gamma_\mu\bar{S}_{p_5}(x'-x)\gamma_\mu, \phi_\nu(x, 0)\;\;D_{k_5}^{(1)}(x-x')\}\end{aligned}$$

---

* To construct $\bar{\varepsilon}$ from ε we should multiply the step function $\sigma(x-x')$ not including x_5. Hence ε is an even function in respect to x_5 as is ε.

$$= -\frac{e^2}{8}\int dk_5\rho(k_5)\sigma(k_5)\int dx'\{[\phi_p^+(x,0),\gamma_\mu S_{-k_5}^{(1)}(x-x')\gamma_\mu\phi_p(x',0)]\,\bar{D}_{k_5}(x-x')$$

$$+[\phi_p^+(x,0),\gamma_\mu\bar{S}_{-k_5}(x-x')\gamma_\mu\phi_p(x',0)]\,D_{k_5}^{(1)}(x-x')$$

$$+[\phi_p^+(x',0)\gamma_\mu S_{k_5}^{(1)}(x'-x)\gamma_\mu,\phi_p(x,0)]\,\bar{D}_{k_5}(x-') \tag{3·4}$$

$$+[\phi_p^+(x',0)\gamma_\mu\bar{S}_{k_5}(x'-x)\gamma_\mu,\phi_p(x,0)]\,D_{k_5}^{(1)}(x-x')\}.$$

$$<H_{1(m),}> = \frac{1}{4}\,[\phi_p^+(x,0),\Pi_p(x,0)] + \frac{1}{4}\,[\Pi_p^+(x,0),\phi_p(x,0)], \tag{3·5}$$

where

$$\Pi_p(x,0) = -\frac{e^2}{2}\int dk_5\rho(k_5)\sigma(k_5)\int dx'\gamma_\mu\Big[\bar{D}_{k_5}(x-x')\Big(\gamma_\lambda\frac{\partial}{\partial x_\lambda}-m-i\gamma_5k_5\Big)\Delta_{k_5}^{(1)}(x-x')$$

$$+D_{k_5}^{(1)}(x-x')\Big(\gamma_\lambda\frac{\partial}{\partial x_\lambda}-m-i\gamma_5k_5\Big)\bar{\Delta}_{k_5}(x-x')\Big]\gamma_\mu\phi_p(x',0)\,dx'.$$

$\Pi_p(x,0)$ can be written in the form

$$\Pi_p(x,0) = -\frac{e^2}{2}\int dk_5\rho(k_5)\sigma(k_5)\int dx' K(x-x')\phi_p(x',0)\,dx'$$

$$= -\frac{e^2}{2}\int dk_5\rho(k_5)\sigma(k_5)\int dp\, e^{ip_\lambda x_\lambda}K(p)\phi_p(p), \tag{3·6}$$

where $\phi_p(p)$ and $K(p)$ are the Fourier transforms of $\phi_p(x,0)$, $K(x)$ respectively;

$$\phi_p(x,0) = \int e^{ip_\lambda x_\lambda}\phi_p(p)\,dp,$$

$$K(x) = \frac{1}{(2\pi)^4}\int e^{ip_\lambda x_\lambda}K(p)\,dp. \tag{3·7}$$

Similarly

$$\Pi_p^+(x,0) = -\frac{e^2}{2}\int dk_5\rho(k_5)\sigma(k_5)\int dp\, e^{-ip_\lambda x_\lambda}\phi_p^+(p)\,K'(p). \tag{3·6′}$$

In (3·6′) $K(p)$ and $K'(p)$ are

$$K(p) = \frac{1}{(2\pi)^3}\int dq\left[\frac{\delta(q_\lambda^2+m^2+k_5^2)}{(p-q)_\lambda^2+k_5^2}+\frac{\delta((p-q)_\lambda^2+k_5^2)}{(q_\lambda^2+m^2+k_5^2)}\right]\gamma_\mu\,[i\gamma_\lambda q_\lambda-m+i\gamma_5k_5]\,\gamma_\mu,$$

$$K'(p) = \frac{1}{(2\pi)^3}\int dq\left[\frac{\delta(q_\lambda^2+m^2+k_5^2)}{(p-q)_\lambda^2+k_5^2}+\frac{\delta((p-q)_\lambda^2+k_5^2)}{(q_\lambda^2+m^2+k_5^2)}\right]\gamma_\mu\,[i\gamma_\lambda q_\lambda-m-i\gamma_5k_5]\,\gamma_\mu. \tag{3·8}$$

Hence $<H_{1(m),0}> = \frac{1}{2}\,\delta m\,[\phi_p^+,\phi_p],$

$$\delta m=\frac{\alpha}{2\pi^2}\int dk_5\rho(k_5)\sigma(k_5)\int dq(i\gamma_\lambda q_\lambda+2m)\left[\frac{\delta(q_\lambda{}^2+m^2+k_5{}^2)}{(p-q)_\lambda{}^2+k_5{}^2}+\frac{\delta((p-q)_\lambda{}^2+k_5{}^2)}{q_\lambda{}^2+m^2+k_5{}^2}\right]. \tag{3·9}$$

This is the same expression as Pauli's one.

§ 4. Photon Self-Energy and Vacuum Polarization

In this section we treat the problem of vacuum polarization and photon self-energy. The calculation is rather complicated but it can be performed in the same line as has been done hitherto by several authors.

When an external field U_λ is present, the change of the energy operator δH is

$$\begin{aligned}<\delta H>_{0,1(0)}&=-<j_\lambda(x,x_5)>_{0,1(0)}U_\lambda(x,x_5)_{1(0)}\\ &=\frac{4e^2}{2\pi}\int dx_5'\int dx'K_{\mu\nu}(x-x',x_5-x_5')[U_\mu(x',x_5')\cdot U_\nu(x,x_5)]_{1(0)},\end{aligned} \tag{4·1}$$

where

$$K_{\mu\nu}=\frac{\partial\bar{E}}{\partial x_\mu}\frac{\partial E^{(1)}}{\partial x_\nu}+\frac{\partial\bar{E}}{\partial x_\nu}\frac{\partial E^{(1)}}{\partial x_\mu}-\delta_{\mu\nu}\left(\frac{\partial\bar{E}}{\partial x_\lambda}\frac{\partial E^{(1)}}{\partial x_\lambda}+m^2\bar{E}E^{(1)}+\frac{\partial\bar{E}}{\partial x_5}\frac{\partial E^{(1)}}{\partial x_5}\right). \tag{4·2}$$

Therefore $\dfrac{\partial K_{\mu\nu}}{\partial x_\nu}$ has the form

$$-\frac{\partial}{\partial x_5}\left(\frac{\partial E}{\partial x_5}\frac{\partial E^{(1)}}{\partial x_\mu}+\frac{\partial E}{\partial x_\mu}\frac{\partial E^{(1)}}{\partial x_5}\right)-\delta(x)f(x_5)\frac{\partial E^{(1)}}{\partial x_\mu}, \tag{4·3}$$

where $f(x_5)=\int e^{ik_5x_5}\rho(k_5)\,dk_5$. The first term gives no contribution. The second term can be written as $-\delta(x)\int dk_5\int dq_5\,e^{ik_5x_5}\rho(k_5-q_5)\sigma(q_5)\dfrac{\partial\Delta^{(1)}_{q_5}}{\partial x_\mu}$ and owing to the independency of gauge function on x_5,* we get integrating over x_5

$$\int\rho(q_5)\,\sigma(q_5)\,dq_5\left[-\delta(x)\frac{\partial\Delta^{(1)}_{q_5}}{\partial x_\mu}\right]. \tag{4·4}$$

This is also the same as Pauli's one.

Hence the photon self-energy term becomes

$$\delta H=4e^2\int dq_5\rho(q_5)\sigma(q_5)\int dx'A_\mu(x',0)A_\nu(x,0)K_{\mu\nu}(x-x',q_5), \tag{4·5}$$

---

* If we take the gauge transformation

$$U_i'(x,x_5)=U_i(x,x_5)-\frac{\partial}{\partial x_i}\Lambda(x,x_5),$$

we get from the assumption $U_5=0$.

$$\frac{\partial}{\partial x_5}\Lambda(x,x_5)=0.$$

This indicates that the gauge-function is independent on x_5.

and

$$\int dq_5 \rho(q_5)\sigma(q_5) K_{\mu\nu}(x, q_5) = \frac{1}{(2\pi)^4} \int e^{ip_\lambda x_\lambda} K_{\mu\nu}(p)\, dp, \tag{4·6}$$

where

$$K_{\mu\nu}(p) = \frac{1}{(2\pi)^3} \int dq_5 \rho(q_5)\sigma(q_5) \int dq \frac{\delta(q_\lambda^2 + m^2 + q_5^2)}{(p-q)_\lambda^2 + m^2 + q_5^2} \times$$

$$\times [-q_\mu(p_\nu - q_\nu) - q_\nu(p_\mu - q_\mu) - \delta_{\mu\nu}\{-q_\lambda(p_\lambda - q_\lambda) + m^2 + q_5^2\}]. \tag{4·7}$$

This has again the regularized form.

§ 5. The Anomalous Moment of the Electron

To discuss the anomalous magnetic moment of an electron, we have only to calculate the change of current operator $\Delta j_\mu(x, x_5)$ where an electron is placed in an external field. After some calculations, we obtain

$$<\Delta j_\mu(x, x_5)>_{1(m),0} = -\frac{ie^2}{2} \frac{1}{(2\pi)^5} \int \rho(k_5)^2 \sigma(k_5)\, dk_5 \times$$

$$\iint d\xi\, d\eta\, \phi_p^+(x+\xi, 0)\gamma^\alpha [\bar{D}_{k_5}(\xi-\eta)\bar{S}_{k_5}(\xi)\gamma^\mu S^{(1)}_{-k_5}(-\eta)$$

$$+ \bar{D}_{k_5}(\xi-\eta) S^{(1)}_{k_5}(\xi)\gamma^\mu \bar{S}_{-k_5}(-\eta)$$

$$+ D^{(1)}_{k_5}(\xi-\eta)\bar{S}_{k_5}(\xi)\gamma^\mu \bar{S}_{-k_5}(-\eta)]\gamma^\alpha \phi_p(x+\eta, 0).$$

To bring this expression to some regularized form, we must put some condition to the function $\rho(k_5)^2\sigma(k_5)$. (See Appendix II.) This circumstance is similar to the case of Pauli's method, and the higher order calculations would necessitate to impose more strict conditions upon $\rho(k_5)$ and $\sigma(k_5)$.

§ 6. Concluding Remarks

It seems that we have shown the equivalence of our formal five dimensional theory to Pauli's regulator at least up to second order approximation. But the vivid perspective is not obtained whether such a method is effective in higher order approximation. The origin of this situation lies in the fact that this equivalence is only in second order, and the difference becomes distinct when the approximation proceeds. The differences are:

1. While Pauli has turned his attention to the square of mass, our method includes essentially the mass itself. The difference does not come into appearence fortunately in second order approximation, but does in higher order. In fact, it is seen that the terms due to this difference arise in the computation of anomalous magnetic moment of the electron.

2. While Pauli's method avoids to perform factorization, our theory shows that in second order approximation the factorization gives the same results. Nevertheless, in higher order approximation this difference becomes distinguished. (See Appendix II.)

In view of these circumstances it must be carefully investigated in detail whether our method is wholly consistent or not.

In our formulation five-dimensional covariance is discarded standing on the viewpoint that we are treating the problems which are essentially four-dimensional, but this point must be furthermore investigated. The principally relating problem is how to formulate the concept of mass state of which we assume *a-priori* in this paper. At least in the present formulation, it is natural to take $k_5=0$ as the ground state, but it is not necessary. Probably, if one could formulate the covariant theory of five-dimensional space throughout, one could success the justification of this concept.

At the end of this paper, we emphasize the fact that one need not tend the auxiliary mass appearing in the intermediate state to infinity in the final results. However, if we don't introduce these limiting process, the results would depend on the form of $\rho(k_5)$ and $\sigma(k_5)$ and then we get different result from the proceeding calculations by many authors. Therefore whether we can determine $\rho(k_5)$ and $\sigma(k_5)$ consistent with these empirical results, e.g. the Lamb-shift and the intrinsic magnetic moment of the electron, or we ought to introduce these limiting process, must be also our future task.

Acknowledgement

In conclusion we should like to express our cordial thanks to Professor M. Kobayasi for his kind interest in this work.

Appendix I.

Since $(2\cdot12)$ and $(2\cdot15)$ contain two time functions, we decompose once again these variables into

$$\overset{(+)}{U}_\mu(x_t)=\overset{(+)}{U}_{P,\mu}(x_t)+\overset{(+)}{U}_{N,\mu}(x_t),$$
$$\overset{(-)}{U}_\mu(x_t)=\overset{(-)}{U}_{P,\mu}(x_t)+\overset{(-)}{U}_{N,\mu}(x_t),$$

and take the Fourier expansion in k_5 space, then the commutators between $\overset{(+)}{A}_{\{{P \atop N}\}}$, $\overset{(-)}{A}_{\{{P \atop N}\}}$, become as follows:

$$[\overset{(+)}{A}_{P,\mu}(x,k_5),\overset{(-)}{A}_{P,\nu}(x',k_5')]=i\delta_{\mu\nu}\delta(k_5+k_5')\frac{\rho(k_5)+\sigma(k_5)}{2\,|\,a(k_5)|^2}\overset{(+)}{D}_{k_5}(x-x'),$$

$$[\overset{(+)}{A}_{N,\mu}(x,{}_5k),\overset{(-)}{A}_{N.\nu}(x',k_5')]=i\delta_{\mu\nu}\delta(k_5+k_5')\frac{\rho(k_5)-\sigma(k_5)}{2\,|\,a(k_5)|^2}\overset{(-)}{D}_{k_5}(x-x').$$

Next, taking the momentum expansions, these commutators can be written as

$$[\overset{(+)}{A}_{P,\mu}(k, k_5), \overset{(-)}{A}_{P,\nu}(k', k_5')]=\delta_{\mu\nu}\delta(\vec{k}-\vec{k}')\,\delta(k_5+k_5')\,\frac{\rho(k_5)+\sigma(k_5)}{2\,|\,a(k_5)|^2},$$

$$[\overset{(+)}{A}_{N,\mu}(k, k_5), \overset{(-)}{A}_{N,\nu}(k', k_5')]=\delta_{\mu\nu}\delta(\vec{k}-\vec{k}')\,\delta(k_5+k_5')\,\frac{\rho(k_5)-\sigma(k_5)}{2\,|\,a(k_5)|^2},$$

Therefore the numbers of particles in these two fields are

$$N_P(k, k_5)=\frac{2\,|\,a(k_5)|^2}{\rho(k_5)+\sigma(k_5)}\,\overset{(+)}{A}_{P,\mu}\overset{(-)}{A}_{P,\mu},$$

$$N_N(k, k_5)=\frac{2\,|\,a(k_5)|^2}{\rho(k_5)-\sigma(k_5)}\,\overset{(+)}{A}_{N,\mu}\overset{(-)}{A}_{N\,\mu}.$$

which imply the positive and negative energy particles respectively.

We can take the similar procedure for the electron field and then obtain the unfamiliar particle field beside the usual one.[5]

Appendix II. Example of Regularization.

In second order approximation, the regularization factor appears as follows:

$$\int\rho(k_5)\sigma(k_5)K(k_5)dk_5. \tag{II–1}$$

If k_5's distribute discretely in the intermediate state, we assume

$$\rho(k_5)=\sum_i\delta_{k_5,\,m_i}\qquad \sigma(k_5)=\sum_i C_i\delta_{k_5,\,m_i}, \tag{II–2}$$

where $m_0=0$ and $C_0=1$, then (II–1) becomes

$$\sum_i C_i\,K(m_i), \tag{II–3}$$

which is equivalent to Stueckelberg-Rivier's procedure.

In higher approximations, taking these assumptions, the regularizations are also equivalent to Pauli's one not using factorization. But if k_5's distribute continuously, the circumstances are not so simple. As an example, we get for the magnetic moment of an electron

$$\int\rho(k_5)^2\sigma(k_5)K(k_5)dk_5, \tag{II–4}$$

and for the case of (II–2) we have

$$\sum_i C_i\,K(m_i),$$

but for the case of continuous distribution, the conditions are not so simple as this.

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On the Decay of Heavy Mesons II*

—Case of the Symmetrical Interaction—**

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§ 1. Introduction

Existence of a heavy meson (τ-meson) having the mass about 700~1000 times that of an electron, and the fact it disintegrates into π-mesons is almost confirmed by recent experiments. For example, G. D. Rochester and C. C. Butler[1] have reported the processes which are considered to be $\tau^{\pm}\rightarrow\pi^{\pm}+\pi^{0}$ and $\tau^{0}\rightarrow\pi^{+}+\pi^{-}$, and C. H. Powell and others[2] have observed the decay process in which a τ-meson disintegrated into three π-mesons. Furthermore, L. Leprince-Ringuet[3] obtained the interesting photograph in which a τ^{-}-meson gave rise to a star and a σ-meson, produced from that star, led to the next star.

Here, the following processes are considered:

$$(\alpha)\quad \tau^{\pm}\rightarrow\pi^{\pm}+\pi^{0} \qquad (\beta)\quad \tau^{0}\rightarrow\pi^{+}+\pi^{-}$$

$$(\gamma)\quad \tau^{\pm}\rightarrow\pi^{\pm}+\tau^{0} \qquad (\delta)\quad \tau^{0}\rightarrow\pi^{0}+\gamma.$$

(α) and (β) are perfectly equal except for charges. (δ) can not be observed by experiments, but we take up the process for the purpose of studying identities occasionally held between two interaction forms of the meson.

As the explanation of the photograph obtained by Powell and others, the following two alternatives may be considered: one of them occurs by the process $\tau^{\pm}\rightarrow\pi^{\pm}+\pi^{+}+\pi^{-}$ and the other is to assume that the successive two processes, $\tau^{\pm}\rightarrow\pi^{\pm}+\tau^{0}$ and $\tau^{0}\rightarrow\pi^{+}+\pi^{-}$ occur.

The former will be treated in part IV, and will not be touched here.

A τ-meson is assumed to be a Boson. As it is considered that τ-mesons interact with nucleons from the photograph of L. Leprince-Ringuet,[3] it seems appropriate that the processes mentioned above occurs through nucleon fields.

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** In the case where electromagnetic field exists, as photons interact with a proton but do not with a neutron, matrix elements of $\tau^{\pm}\rightarrow\pi^{\pm}+\gamma$ becomes unsymmetrical with regard to nucleons. In this paper, these situations do not occur. We call this "*symmetrical interaction.*" Former cases are called "*unsymmetrical interaction.*"

Firstly, constructing the equation of motion for the process in which a heavy meson decays into lighter ones via nucleons, the decay life-time of a τ-meson into π-meson was calculated, and the selection rule for possible transition processes was researched. Next, the equivalency between two different interaction forms of scalar and pseudoscalar meson was studied* and found that there are many cases where Nelson$^{4)}$ and Dyson's$^{5)}$ arguments do not hold. Further, it is studied how we use regulator proposed by Pauli and Villars$^{6)}$ for the divergency, non-gauge covariancy and the equivalence theorem, on which was already delivered in our paper I.$^{7)}$ The calculation was done by using Tomonaga-Schwinger's formalism.

§ 2. Method of Calculation and the General Hamiltonian Form

The fundamental equation for mesons, nucleons and photons is the following:

$$i\hbar c\frac{\delta\Psi[\sigma]}{\delta\sigma(x)}=H(x)\Psi[\sigma] \tag{1}$$

where $H(x)$ is the Hamiltonian density of the system consisting of a initial Boson, final Bosons, and nucleons. $H(x)$ is divided into two parts $H_A(=H_1)$ and $H_B=H_2+H_3$. H_A is the Hamiltonian concerning to a initial Boson and H_2 and H_3 are of final Bosons. Each has the form $g\bar{\psi}O(\tau_{NP}+\tau_{PN})\psi\xi+f\bar{\psi}O\tau\psi\eta$, where g and f are coupling constants of a Boson with nucleon, ψ is the wave function of nucleon ($\bar{\psi}=\psi^*\gamma_4$), and ξ and η are the wave functions or their derivatives of Bosons. τ_{NP}, τ_{PN}, and τ are the isotopic spin matrices of nucleons, $\tau_{NP}=(\tau_1+i\tau_2)/2$, $\tau_{PN}=(\tau_1-i\tau_2)/2$ and τ is τ_3 for symmetrical theory and equal 1 for neutral theory.

O is the Dirac matrix. Then we perform the canonical transformation

$$\Psi[\sigma]=U[\sigma]\Psi'[\sigma] \tag{2}$$

where $U[\sigma]$ satisfies the relation

$$i\hbar c\frac{\delta U[\sigma]}{\delta\sigma(x)}=H_B(x)U[\sigma] \tag{3}$$

By this transformation we obtain

$$i\hbar c\frac{\delta\Psi'[\sigma]}{\delta\sigma(x)}=U^{-1}[\sigma]H_A(x)U[\sigma]\Psi'[\sigma] \tag{4}$$

and further, when the relation

$$U^{-1}H_AU=H_A-\frac{i}{\hbar c}\int_{-\infty}^{\sigma}d\omega'[H_A,H_B']-\frac{1}{(\hbar c)^2}\int_{-\infty}^{\sigma}d\omega'\int_{-\infty}^{\sigma'}d\omega''[[H_A,H_B'],H_B'']+\cdots\cdots$$

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* Generally, the identities hold also in the case $\tau^\pm\rightarrow\pi^\pm+\gamma$. These cases will be treated in Part III.

is used, it is sufficient to calculate the expression

$$H=-\frac{1}{(\hbar c)^2}\int_{-\infty}^{\sigma}d\omega'\int_{-\infty}^{\sigma'}d\omega''[[H_A, H_B'], H_B''] \tag{5}$$

for the processes considered. According to Schwinger, we get the following Hamiltonian density,

$$\frac{Ggf}{2(\hbar c)^2}\iint_{-\infty}^{\infty}\mathrm{Sp}\,\{S^{(1)}(x''-x)\,O_1\,\bar{S}(x-x')O_2\bar{S}(x'-x'')\,O_3$$

$$+\varepsilon S^{(1)}(x-x'')\,O_3\bar{S}(x''-x')\,O_2\bar{S}(x'-x)\,O_1+\bar{S}(x''-x)\,O_1S^{(1)}(x-x')\,O_2\bar{S}(x'-x'')\,O_3$$

$$+\varepsilon\bar{S}(x-x'')\,O_3S^{(1)}(x''-x')\,O_2\bar{S}(x'-x)\,O_1+\bar{S}(x''-x)\,O_1\bar{S}(x-x')\,O_2S^{(1)}(x'-x'')\,O_3$$

$$+\varepsilon\bar{S}(x-x'')\,O_3\bar{S}(x''-x')\,O_2S^{(1)}(x'-x)\,O_1\}\times U_1V_2'\,W_3''\,d\omega'\,d\omega'' \tag{6}$$

where U_1, V_2, and W_3 are wave functions or their derivatives of Bosons, U_1 of initial and V_2 and W_3 of final, and O_1, O_2 and O_3 are Dirac matrices. ε equals to 1 for neutral theory and -1 for symmetrical theory.

Applying the Schwinger's expressions for the $\bar{S}$ and $S^{(1)}$ appeared in (6), H can be written in the following form,

$$H=H_{\mathrm{I}}+H_{\mathrm{II}}+H_{\mathrm{III}}+H_{\mathrm{IV}}.$$

$$H_{\mathrm{I}}=-\frac{\varkappa^3Ggf}{8\,(\hbar c)^2\,(2\pi)^{12}}\,\mathrm{Sp}(O_1O_2O_3+\varepsilon\,O_3O_2O_1)\iint_{-\infty}^{\infty}[\int(dk^0)\,(dk'^0)\,(dk''^0)\times$$

$$\times\int_{-\infty}^{\infty}da\,db\,dc\,S\,e(\xi,\eta)e(k_\nu''^2)e(\varkappa^2)\,e(R)]\,U_1(x)\,V_2(x+\xi)\,W_3(x-\eta)\,d\xi\,d\eta \tag{8}$$

$$H_{\mathrm{II}}=-\frac{i\varkappa^2Ggf}{8\,(\hbar c)^2\,(2\pi)^{12}}\iint_{-\infty}^{\infty}[\int(dk^0)\,(dk'^0)\,(dk''^0)\int_{-\infty}^{\infty}da\,db\,dc\,S\,\{\mathrm{Sp}(O_1O_2\gamma_\alpha O_3-\varepsilon\,O_3\gamma_\alpha O_2O_1)\,L_\alpha$$

$$+\mathrm{Sp}\,(O_1\gamma_\alpha O_2O_3-\varepsilon\,O_3O_2\gamma_\alpha O_1)\,M_\alpha+\mathrm{Sp}(\gamma_\alpha O_1O_2O_3-\varepsilon\,O_3O_2O_1\gamma_\alpha)\,N_\alpha\}\times$$

$$\times\,e(\xi,\eta)\,e(k_\nu''^{02})\,e(\varkappa^2)\,e(R)]\,U_1(x)\,V_2(x+\xi)\,W_3(x-\eta)\,d\xi\,d\eta \tag{9}$$

$$H_{\mathrm{III}}=H_{\mathrm{III\cdot A}}+H_{\mathrm{III\cdot B}}\,,\quad H_{\mathrm{IV}}=H_{\mathrm{IV\cdot A}}+H_{\mathrm{IV\cdot B}}$$

$$H_{\mathrm{III\cdot A}}=-\frac{\varkappa Ggf}{8\,(\hbar c)^2\,(2\pi)^{12}}\iint_{-\infty}^{\infty}[\int(dk^0)\,(dk'^0)\,(dk''^0)\int_{-\infty}^{\infty}da\,db\,dc\,S\,\{\mathrm{Sp}(O_1\gamma_\alpha O_2\gamma_\beta O_3$$

$$+\varepsilon O_3\gamma_\beta O_2\gamma_\alpha O_1)\,M_\alpha L_\beta+\mathrm{Sp}(\gamma_\beta O_1O_2\gamma_\alpha O_3+\varepsilon\,O_3\gamma_\alpha O_2O_1\gamma_\beta)\;N_\beta L_\alpha+\mathrm{Sp}(\gamma_\beta O_1\gamma_\alpha O_2O_3$$

$$+\varepsilon\,O_3O_2\gamma_\alpha O_1\gamma_\beta)\,M_\alpha N_\beta\}e(\xi,\eta)e(k_\nu''^{02})e(\varkappa^2)e(R)]\,U_1(x)\,V_2(x+\xi)\,W_3(x-\eta)\,d\xi\,d\eta \tag{10}$$

$$H_{\mathrm{III\cdot B}}=-\frac{\varkappa Ggf}{8\,(\hbar c)^2\,(2\pi)^{12}}\,\mathrm{Sp}(O_1\gamma_\alpha O_2\gamma_\alpha O_3+\varepsilon\,O_3\gamma_\alpha O_2\gamma_\alpha O_1+\gamma_\alpha O_1\gamma_\alpha O_2O_3+\varepsilon\,O_3O_2\gamma_\alpha O_1\gamma_\alpha$$

$$+\gamma_\alpha O_1 O_2 \gamma_\alpha O_3 + \varepsilon O_3 \gamma_\alpha O_2 O_1 \gamma_\alpha) \iint_{-\infty}^{\infty} [\int (dk^0)(dk'^0)(dk''^0) \int_{-\infty}^{\infty} da\,db\,dc\, S k_\beta''^{02}\, e(\xi,\eta) \times$$

$$\times e(k_\nu''^{02}) e(\boldsymbol{x}^2) e(R)]\, U_1(x)\, V_2(x+\xi)\, W_3(x-\eta))\, d\xi\, d\eta \qquad (11)$$

where the suffix β of $k_\beta''^{02}$ does not mean summation.

$$H_{\text{IV}\cdot\text{A}} = \frac{iGgf}{8(hc)^2(2\pi)^{12}} \text{Sp}(\gamma_\beta O_1 \gamma_\alpha O_2 \gamma_\gamma O_3 - \varepsilon O_3 \gamma_\gamma O_2 \gamma_\alpha O_1 \gamma_\beta) \iiint_{-\infty}^{\infty} [\int (dk^0)(dk'^0)(dk''^0)$$

$$\times \int_{-\infty}^{\infty} da\,db\,dc\, S M_\alpha N_\beta L_\gamma e(\xi,\eta) e(k_\nu''^{02}) e(x^2) e(R)]\, U_1(x)\, V_2(x+\xi)\, W_3(x-\eta)\, d\xi\, d\eta$$
$$(12)$$

$$H_{\text{IV}\cdot\text{B}} = \frac{iGgf}{8(\hbar c)^2(2\pi)^{12}} \iint_{-\infty} [\int (dk^0)(dk'^0)(dk''^0) \int_{-\infty}^{\infty} da\,db\,dc\, S\, \{\text{Sp}(\gamma_\alpha O_1 \gamma_\alpha O_2 \gamma_\beta O_3$$

$$- \varepsilon O_3 \gamma_\beta O_2 \gamma_\alpha O_1 \gamma_\alpha) L_\beta + \text{Sp}(O_1 \gamma_\alpha O_2 \gamma_\alpha O_3 \gamma_\beta - \varepsilon \gamma_\beta O_3 \gamma_\alpha O_2 \gamma_\alpha O_1) N_\beta$$

$$+ \text{Sp}(O_1 \gamma_\beta O_2 \gamma_\alpha O_3 \gamma_\alpha - \varepsilon \gamma_\alpha O_3 \gamma_\alpha O_2 \gamma_\beta O_1) M_\beta\} k_\gamma''^{02} e(\xi,\eta) e(k_\nu''^{02}) e(x^2) e(R)] \times$$

$$U_1(x)\, V_2(x+\xi)\, W_3(x-\eta)\, d\xi\, d\eta \qquad (13)$$

where the suffix γ of $k_\gamma''^{02}$ does not mean summation.

In these expressions the following abbreviations are used

$$S = \left(\frac{a}{|a|} + \frac{b}{|b|}\right)\left(\frac{c}{|c|} + \frac{a+b}{|a+b|}\right), \qquad e(\xi,\eta) = \exp i\,(k_\nu^0 \xi_\nu + k_\nu'^0 \eta_\nu)$$

$$e(\boldsymbol{x}^2) = \exp i\,(a+b+c)\,\boldsymbol{x}^2, \qquad e(k_\nu''^{02}) = \exp i\,(a+b+c)\, k_\nu''^{02}$$

$$e(R) = \exp\,(-i)\{a\boldsymbol{x}_2^2 + b\boldsymbol{x}_3^2 + (ak_\nu^0 + bk_\nu'^0)^2/(a+b+c)\}$$

$$L_\alpha = (ak_\alpha^0 + bk_\alpha'^0)/(a+b+c), \quad M_\alpha = L_\alpha - k_\alpha^0, \quad N_\alpha = L_\alpha - k_\alpha'^0.$$

$\boldsymbol{x}$ is the reciprocal Compton wave length of a nucleon and $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$ are that of V_2 and W_3 respectively. ξ and η stand for $x'-x$ and $x-x''$.

These terms are classified into two sets of different kinds, say, H_{I}, H_{III} and H_{II}, H_{IV} according to the number of matrices γ_μ contained in O_1, O_2 and O_3, for the spur of the product of odd number of γ_μ vanishes. In the case of a γ-decay of a charged meson, according to its non-symmetric character of the coupling with nucleons, only such terms remain that the order of O_1, O_2 and O_3 in the spur is $O_3 O_2 O_1 \cdot H_{\text{III}\cdot\text{B}}$ and $H_{\text{IV}\cdot\text{B}}$ are the terms containing logarithmically divergences.

It is necessary to notice that H_{I} and H_{III} contains the odd powers of $\boldsymbol{x}$ and H_{II}, H_{IV} even powers of $\boldsymbol{x}$. We use the symbol $\boldsymbol{x}^1$] for the former and [$\boldsymbol{x}^0$] for the latter.

§ 3. Identities I.

As stated in § 1, Dyson and Nelson have pointed out the fact that in the case of pseudoscalar meson, its pseudoscalar interaction and its pseudovector one are equivalent, and in the case of scalar meson, its vector interaction vanishes at least to the order of g^2. We checked these statements for our processes. Firstly, we shall consider about the initial Boson, and use (6) for the Hamiltonian causing the processes.

(i) Cases where initial Boson is of scalar type.

Denote the Hamiltonian for its scalar interaction with $H_1(s)$ and for its vector one with $H_2(s)$. For the cases of pseudoscalar Bosons, similar notations are used; Then we easily find that

$$H_2(s) = \frac{i}{2}\left(\frac{1}{\hbar c}\right)^2 \frac{G_2}{\kappa_\tau} gf \frac{\partial}{\partial x_\mu} \left[\iint_{-\infty}^{\infty} U V' W'' d\omega' d\omega'' \,\mathrm{Sp}\,(\cdots\cdots)\right] + \frac{\kappa G_2}{\kappa_\tau G_1} G_1 [H_1(s) - H_1(s)] + B \tag{14}$$

where $B=0$ for neutral theory

$$= -i \frac{G_2 gf}{(\hbar c)^2 \kappa_\tau} \Big[\int_{-\infty}^{\infty} U V_2'' W_3 d\omega' \,\mathrm{Sp}\,\{S^{(1)}(x-x') O_2 \bar{S}(x'-x) O_3$$

$$+ \bar{S}(x-x') O_2 S^{(1)}(x'-x) O_3\} - \int_{-\infty}^{\infty} U V_2 W_3' d\omega' \,\mathrm{Sp}\,\{S^{(1)}(x-x') O_3 \bar{S}(x'-x) O_2$$

$$+ \bar{S}(x-x') O_3 S^{(1)}(x'-x) O_2\}\Big] \quad \text{for symmetrical theory.}$$

$Sp\,(\cdots\cdots)$ contains the factor γ_μ and the first term of the right hand side is the four dimensional divergence. Since we treat only the cases where the law of conservation of energy and momentum holds, this term vanishes identically. Namely, for the neutral theory, Dyson's statement is always established. For the symmetrical theory, B does not vanish generally.

(ii) The case where initial Boson is of pseudoscalar type. $H_2(ps)$ can, be written

$$H_2(ps) = -\frac{i}{2}\left(\frac{1}{\hbar c}\right)^2 \frac{G_2}{\kappa_\tau} gf \frac{\partial}{\partial x_\mu} \left[\iint_{-\infty}^{\infty} U V' W'' dw' dw'' \,\mathrm{Sp}(\cdots\cdots)\right] + \frac{2\kappa G_2}{\kappa_\tau G_1} G_1 H_1(ps) + B \tag{15}$$

where

$$B = -\frac{i}{2} \frac{G_2 gf}{(\hbar c)^2 \kappa_\tau} \Big[\int_{-\infty}^{\infty} U V_2'' W_3 d\omega' \,\mathrm{Sp}\,\{S^{(1)}(x-x') O_2 S(x'-x) O_3 \gamma_5 + \varepsilon S^{(1)}(x-x') O_2 \times$$

$$\times \bar{S}(x'-x) \gamma_5 O_3 + \bar{S}(x-x') O_2 S^{(1)}(x'-x) O_3 \gamma_5 + \varepsilon \bar{S}(x-x') O_2 S^{(1)}(x'-x) \gamma_5 O_3\}$$

$$+\int_{-\infty}^{\infty} UV_2W_3'\,d\omega'\,\mathrm{Sp}\{S^{(1)}(x-x')O_3\bar{S}(x'-x)\gamma_5O_2+\varepsilon S^{(1)}(x-x')O_3\bar{S}(x'-x)O_2\gamma_5$$
$$+S(x-x')O_3S^{(1)}(x'-x)\gamma_5O_2+\varepsilon\bar{S}(x-x')O_3S^{(1)}(x'-x)O_2\gamma_5\}]$$

In these calculations we have used the relations such as $(\gamma_\mu\partial_\mu+\varkappa)S^{(1)}(x-x')=0$, $(\gamma_\mu\partial_\mu+\varkappa)\bar{S}(x-x')=-\delta(x-x')$ and so on, and B is the term caused from the part $\delta(x-x')$. When B vanishes, if we equate C_1 to $2\varkappa G_2/\varkappa_\tau$, then the identity

$$H_2(ps)=H_1(ps)$$

holds. But as B does not always vanish, we may say that there are considerably many cases where equivalence theorem does not hold.

About the final Bosons, if we take the Hamiltonian density of the final Boson V_2 considered as H_A in (5), the same expression as the initial Boson is obtained. For example, in the case of a pseudoscalar meson,

$$H_2(ps)=\frac{i}{2}\left(\frac{1}{\hbar c}\right)^2\frac{g_2}{\varkappa_\tau}\,Gf\,\frac{\partial}{\partial x_\mu}\iint_{-\infty}^{\infty}V_2U_1'W_3''\,d\omega'\,d\omega''\,\mathrm{Sp}(\cdots\cdots)+H_1(ps)+B \quad (16)$$

and the first term of the right hand side vanishes by the same reason as mentioned above*. Therefore, if $B=0$, the identity relation holds between pseudoscalar and pseudovector coupling.

As was seen from (14) and (16), these relations have the form

$$H_2=\varkappa(H_1\pm H_1)+B \qquad (17)$$

where H_1 stand for the expression $(G_2/\varkappa_\tau)H$. The significance of (17) with regard to the regulating method is stated in part I, in spite of that, we will repeat it briefly in § 5.

§ 4. Selection Rule and Tables

In calculating the transition processes of the 3rd order, making use of Hamiltonians from (8) to (13), it is found that there is a certain selection rule for the forbidden processes. For example, in the case of $\tau^\pm\rightarrow\pi^\pm+\pi^0$, assuming that $\pi^\pm$ and π^0 obey wave equations of the same type, we have 128 combinations in all. Among them, many terms vanish by taking into account the characters of spurs, forms of wave functions raised from the differentiation of Δ and $\Delta^{(1)}$. Investigation of these vanishing terms shows that if the spur of H_{I} vanishes, then the spurs of H_{III} vanish and if the one of the spurs of H_{II} vanish, then the rest of H_{II} and the spurs of H_{IV} vanish.

When spur of H_{I} vanishes identically owing to the existence of γ_5 (say, the case of the form $Sp\,(\gamma_5\gamma_\mu\gamma_\nu))$, it is necessary to examine one of the spurs of $H_{\mathrm{III}\cdot A}$.

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* One example of this will be given in Appendix 1.

This fact is of very interesting and we have reached the following selection rules held for the forbidden processes. Namely,

$$[x^1]\quad Sp(O_1O_2O_3+\varepsilon O_3O_2O_1)\,U_1V_2W_3=0 \quad \textit{forbidden}$$
$$\neq 0 \quad \textit{allowed} \tag{18}$$

$$[x^0]\quad Sp(O_1O_2\gamma_\alpha O_3-\varepsilon O_3\gamma_\alpha O_2O_1)\,U_1\,\partial/\partial x_\alpha\,(V_2W_3)=0 \quad \textit{forbidden}$$
$$\neq 0 \quad \textit{allowed} \tag{19}$$

In the cases where (18) vanish identically owing to γ_5, then

$$[x^1]\quad Sp(O_1\gamma_\alpha O_2\gamma_\beta O_3+\varepsilon O_3\gamma_\beta O_2\gamma_\alpha O_1)\,U_1\,\partial^2/\partial x_\alpha\partial x_\beta\,(V_2W_3)=0 \quad \textit{forbidden} \tag{20}$$
$$\neq 0 \quad \textit{allowed}$$

This rule seems to hold usually for the third order processes of such type that the Boson interact symmetrically with nucleons. Together with the identities stated in § 3, this rule indicates us the forbidden processes almost perfectly.*

The results of adopting these two relations are listed in Table I, II, III. Table I is for the processes (α) and (β), that is, $\tau^\pm\rightarrow\pi^\pm+\pi^0$. Table II is for ($\gamma$), that is, $\tau^\pm\rightarrow\pi^\pm+\tau^0$. Table III is for ($\delta$), namely $\tau^0\rightarrow\pi^0+\gamma$. In these tables $\mathfrak{G}$ and F stand for the coupling constants of a τ-meson with nucleons, g and f of a π-meson, and e of a photon. And, for example, in the case of a pseudoscalar meson, suffix 1 stands for its pseudoscalar coupling and suffix 2 its pseudovector one. And so in other cases. S means that the term is allowed for symmetrical theory ($\tau=\tau_3$) and N for neutral one ($\tau=1$).

It is easily seen that S and N are mutually exclusive. In Table III, this distinction is not necessary, since the only allowable choice is $\tau=\tau_p$.

Div and *Fin* are abbreviations of "*containing divergences*" and "*containing no divergences*" respectively. Of course, divergences are of the order of logari-

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* It can be proved that the relation

$$Sp\,(\gamma_\alpha\gamma_\beta\cdots\cdots\gamma_\mu\gamma_\nu)=Sp\,(\gamma_\nu\gamma_\mu\cdots\cdots\gamma_\beta\gamma_\alpha)$$

holds in our case. Since the orders of matrices containing in the selection rules are opposite in the 1st and the 2nd terms except the order of matrices involved in O_1, O_2 and O_3, when the order of γ-matrices included in O_1, O_2 and O_3 are interchanged in the 2nd term of the rules, then the 2nd term becomes equivalent to the first but for sign. As the O_i whose sign is altered by conversion of the order of its factors is $\gamma_5\gamma_\mu$, $\gamma_\mu\gamma_\nu$, and $\gamma_5\gamma_\mu\gamma_\nu$, therefore, the 2nd term alters its sign when the number of $\gamma_5\gamma_\mu$, $\gamma_\mu\gamma_\nu$, $\gamma_5\gamma_\mu\gamma_\nu$, we call this number "$K$", is odd and does not when K is even, if we make the order of γ-matrices of the second term opposite to the first.

Therefore, we may say the selection rules more simply as follows:

$[K^1]$ *is forbidden when* K *is odd for* $\varepsilon=1$ *and* K *is even for* $\varepsilon=-1$.

$[K^0]$ *is forbidden when* K *is even for* $\varepsilon=1$ *and* K *is odd for* $\varepsilon=-1$.

However, as is easily seen, this rule is less satisfactory than the above, and is, so to speak, the necessary condition and not the sufficient one.

thmic. Symbol * denotes that it vanishes by identity relation (this is, the case of $B=0$).

In the case of $\tau^\pm\to\pi^\pm+\gamma$, the rule mentioned above does not hold because of the absence of symmetry. Symbol $\circ$ denotes that the identity relations hold between those terms.

Table I. $\tau^\pm\to\pi^\pm+\pi^0$ $(\tau^0\to\pi^++\pi^-)$

	$G_1g_1f_1$	$G_1g_1f_2$	$G_1g_2f_1$	$G_1g_2f_2$	$G_2g_1f_1$	$G_2g_1f_2$	$G_2g_2f_1$	$G_2g_2f_2$
$s\to s+s$	ND (1)	SD (0)	SD (0)	—*	SD (0)	—*	—*	SD (0)
$s\to ps+ps$	ND (1)	ND (0)	ND (0)	ND (1)	SD (0)	SD (1)	SD (1)	SD (0)
$ps\to s+s$	—	—	—	—	—	—	—	—
$ps\to ps+ps$	—	—	—	—	—	—	—	—
$s\to v+v$	NF (1)	ND (0)	ND (0)	ND (1)	SD (0)	SD (1)	SD (1)	SD (0)
$s\to pv+pv$	ND (1)	SD (0)	SD (0)	ND (1)	SD (0)	—*	—*	SD (0)
$ps\to v+v$	NF (1)$_\circ$	ND (0)	ND (0)	ND (1)	NF (0)$_\circ$	NF (1)	NF (1)	ND (0)
$ps\to pv+pv$	NF (1)$_\circ$	SD (0)	SD (0)	ND (1)	NF (0)$_\circ$	SD (1)	SD (1)	ND (0)
$v\to s+s$	SD (0)	—*	—*	SD (0)	—	—*	—*	SD (1)
$v\to ps+ps$	SD (0)	SD (1)	SD (1)	SD (0)	—	SD (0)	SD (0)	SD (1)
$pv\to s+s$	—	—	—	—	SF (1)	—*	—*	SD (1)
$pv\to ps+ps$	—	—	—	—	SF (1)	SD (0)	SD (0)	SD (1)
$v\to v+v$	SD (0)	SD (1)	SD (1)	SD (0)	SD (1)	SD (0)	SD (0)	SF (1)
$v\to pv+pv$	SD (0)	NF (1)	NF (1)	SD (0)	SD (1)	ND (0)	ND (0)	SF (1)
$pv\to v+v$	NF (0)	NF (1)	NF (1)	ND (0)	SD (1)	SD (0)	SD (0)	SF (1)
$pv\to pv+pv$	NF (0)	NF (1)	SD (1)	ND (0)	SD (1)	ND (0)	ND (0)	SF (1)

Table II. $\tau^\pm\to\pi^\pm+\tau^0$

$s\to s+s$	ND (1)	SD (0)	SD (0)	—*	SD (0)	—*	—*	SD (0)
$s\to ps+s$	—	—	—	—	—	—	—	—
$s\to v+s$	SD (0)	—*	—	—*	—*	SD (0)	—*	SD (1)
$s\to pv+s$	—	—	—	—	—*	—*	—*	ND (1)
$ps\to s+ps$	ND (1)	ND (0)	SD (0)	SD (1)	ND (0)	ND (1)	SD (1)	SD (0)
$ps\to ps+ps$	—	—	—	—	—	—	—	—
$ps\to v+ps$	SD (0)	SD (1)	—	—*	SD (1)	SD (0)	—*	SD (1)
$ps\to pv+ps$	—	—	—	—	—*	—*	—*	SD (1)
$v\to s+v$	NF (1)	ND (0)	SD (0)	SD (1)	ND (0)	ND (1)	SD (1)	SD (0)
$v\to ps+v$	NF (1)	ND (0)	NF (0)	NF (1)	ND (0)	ND (1)	NF (1)	ND (0)
$v\to v+v$	SD (0)	SD (1)	SD (1)	SD (0)	SD (1)	SD (0)	SD (0)	SF (1)
$v\to pv+v$	NF (0)	NF (1)	SD (1)	SD (0)	NF (1)	ND (0)	SD (0)	SF (1)
$pv\to s+pv$	ND (1)	SD (0)	SD (0)	—*	SD (0)	ND (1)	—*	SD (0)
$pv\to ps+pv$	NF (1)	SD (0)	NF (0)	SD (1)	SD (0)	ND (1)	SD (1)	ND (0)
$pv\to v+pv$	SD (0)	NF (1)	SD (1)	ND (0)	NF (1)	SD (0)	ND (0)	SF (1)
$pv\to pv+pv$	NF (0)	SD (1)	SD (1)	ND (0)	SD (1)	ND (0)	ND (0)	SF (1)

Table III. $\tau^0 \rightarrow \pi^0 + \gamma$

	$G_1 g_1 e$	$G_1 g_2 e$	$G_2 g_1 e$	$G_2 g_2 e$		$G_1 g_1 e$	$G_1 g_2 e$	$G_2 g_1 e$	$G_2 g_2 e$
$s \rightarrow s + \gamma$	—	—*	—	—	$v \rightarrow s + \gamma$	F (1)	—	D (0)	—
$s \rightarrow ps + \gamma$	—	—	—*	—*	$v \rightarrow ps + \gamma$	F (1)	F (0)	D (0)	F (1)
$ps \rightarrow s + \gamma$	—	—	—	—*	$pv \rightarrow s + \gamma$	—	—*	D (0)	—
$ps \rightarrow ps + \gamma$	—	—	—	—	$pv \rightarrow ps + \gamma$	—	—	D (0)	F (1)
$s \rightarrow v + \gamma$	F (1)	D (0)	—	—	$v \rightarrow v + \gamma$	—	—	—	—
$s \rightarrow pv + \gamma$	—	D (0)	—*	—	$v \rightarrow pv + \gamma$	F (0)	—	F (1)	—
$ps \rightarrow v + \gamma$	F (1)	D (0)	F (0)	F (1)	$pv \rightarrow v + \gamma$	F (0)	F (1)	—	—
$ps \rightarrow pv + \gamma$	—	D (0)	—	F (1)	$pv \rightarrow pv + \gamma$	—	F (1)	F (1)	—

(Notations) In these Tables the following abbreviations are used:

S D .... S (Div) S F .... S (Fin)
N D .... N (Div) N F .... N (Fin)
(1) .... $[x^1]$ (0) .... $[x^0]$

§ 5. Identities II and Regulator.

There is no need to consider the problem concerning with gauge covariant except in the case (δ), but if it is remembered that the non-gauge covariant term appeared in the problem of the γ-decay of a neutretto has its origin in the pathological nature of Δ-function, it is consistent to use the regulator removing the non-gauge covariant term there also in our cases. Further, we may adopt the identical relations as one of the guiding principles, that is, because of proving formally of equivalency before the direct calculation in the case of scalar and pseudoscalar meson, the following standpoint may be employed, namely, the terms which do not satisfy the identity must be removed (of course, in the case $B=0$). The suitability of these principles are yet more or less questionable, because as was shown in the previous paper[7], the proof of equivalence theorem is not perfect owing to the pathological nature of Δ-function. In this section, we will explain these by some examples. It is remarkable that in the case (δ), identities always hold as long as the scalar and pseudoscalar mesons are concerned. Its proof is very easy and will be not touched. (The conditions for regulator to remove non-gauge convariant terms are used in this proof).

When $B=0$, as shown in the Tables, there are two different kinds of equivalences (we will exemplify about a pseudoscalar meson), one of them

$$[x^0]=2x\,[x^1] \tag{21}$$

and others

$$[x^1]=2x\,[x^0] \tag{22}$$

As mentioned in part I, there exists some ambiguity in the application of regulating method. Here we will use the relation $H_2=2xH_1$ or $G_1=(2x/x_\tau)\,G_2$ for letting the equivalence theorem hold.

1. Identities of the type $[x^0]=2x\,[x^1]$

Equivalence theorems which held in the γ-decay of a neutretto, viz. $\pi^0\rightarrow\gamma+\gamma$, and also some of them held in the case of $\tau^\pm\rightarrow\pi^\pm+\pi^0$, $\tau^\pm\rightarrow\pi^\pm+\tau^0$, and $\tau^0\rightarrow\pi^0+\gamma$ belong to this type. One of the examples is previously stated in other places,$^{2)}$ and it is shown that there is a term that does not satisfy the identity relation even though it converges, but, is probably ambiguous, and has a gauge covariant form, and removal of which requires the further condition $\int(\rho(x)\sqrt{x})dx=0$. The same situation is shown in the case $ps\rightarrow\tau+\tau(\tau^\pm\rightarrow\pi^\pm+\pi^0)$ when $G_1g_1f_1$ and $G_2g_1f_1$ are calculated (neutral theory), and seems to be the common nature of the identities of this type. The latter example will be given in Appendix 2.

2) Identities of the type $[x^1]=2x\,[x^0]$

This type of identity is discovered first in the case $\tau^0\rightarrow\pi^0+\gamma$ (Table III).

For example, we will pick up the cases G_1g_2e and G_2g_2e of $ps\rightarrow\tau+\gamma$.

As the transitions treated here are, of course, the cases where the law of conservation of energy and momentum hold, we may replace $\Box^2=\partial^2/\partial x_\mu^2$ by x_τ^2 and $2\partial_\nu V_\alpha\partial_\nu A_3$, $2\partial_\nu U\partial_\nu V_\alpha$, and $2\partial_\nu U\partial_\nu A_3$ by $(x_\tau^2-x_\pi^2)V_\alpha A_3$, $-(x_\tau^2+x_\pi^2)UV_\alpha$ and $(x_\pi^2-x_\tau^2)UA_3$ respectively. Applying these relations, $H_2(ps)$ and $H_1(ps)$ can be written in the following forms,

$$H_1(ps)=\frac{G_1g_2e}{4^2(2\pi)^2(\hbar c)^2}Sp(\gamma_5\gamma_\rho\gamma_\tau\gamma_\alpha\gamma_\mu)\varphi\chi_{\rho\sigma}F_{\alpha\mu}\left\{\int_0^\infty\frac{dw}{w}\cos w+\frac{1}{6}\right.$$
$$+\frac{1}{4}\frac{x_\pi^2}{x^2}\int_1^\infty\frac{du}{u^6}\frac{2-u}{1+\frac{1-u}{u^2}\frac{x_\pi^2}{x^2}}-\frac{1}{8}\int_{-1}^1 dv\int_1^\infty\frac{du}{u^6}\times$$
$$\left.\times\frac{(1-u)(4+8u-6v-v^2+8uv)\frac{x_\pi^2}{x^2}-(2-2v+2u+4v^2-v^3-2uv^2)\frac{x_\tau^2}{x^2}}{1+\frac{(1+v)(1-u)}{2u^2}\frac{x_\pi^2}{x^2}-\frac{1-v^2}{4u^2}\frac{x_\tau^2}{x^2}}\right\}\tag{23}$$

$$H_2(ps)=-\frac{xG_2g_2e}{8(2\pi)^2x_\tau(\hbar c)^2}Sp(\gamma_5\gamma_\rho\gamma_\sigma\gamma_\alpha\gamma_\mu)\left\{3A_\mu\chi_{\rho\sigma}\partial_\alpha\varphi+\frac{1}{2}\varphi\chi_{\rho\sigma}F_{\alpha\mu}\int_{-1}^1 dv\int_1^\infty\frac{du}{u^5}\times\right.$$
$$\times\left[(1-v^2+u-2u^2)\frac{x_\pi^2}{x^2}-(1-v^2+u)\frac{x_\tau^2}{x^2}\right]-\frac{i}{4}\varphi\chi_{\rho\sigma}F_{\alpha\mu}\int_{-1}^1 dv\int_1^\infty\frac{du}{u^7}\times$$
$$\left.\times\frac{\left[(1+v)(1-u)\frac{x_\pi^2}{x^2}-\frac{1-v^2}{2}\frac{x_\tau^2}{x^2}\right]\left[(1-v^2+u-uv-2u^2)\frac{x_\pi^2}{x^2}-(1-v^2+u-uv)\frac{x_\tau^2}{x^2}\right]}{1+\frac{(1+v)(1-u)}{2u^2}\frac{x_\pi^2}{x^2}-\frac{1-v^2}{4u^2}\frac{x_\tau^2}{x^2}}\right\}$$
$$\tag{24}$$

As the first and the second terms of $H_1(ps)$ are dropped out by $\int\rho(x)\log|x|\,dx=0$ and $\int\rho(x)dx=0$ and the first term of $H_2(ps)$ by $\int\sqrt{x}\rho(x)dx=0$, it is found that equivalency holds between the terms proportional to $\varkappa_\pi^2/\varkappa^2$ and $\varkappa_\tau^2/\varkappa^2$ of $H_1(ps)$ and $H_2(ps)$.

$$H_1(ps)=\frac{G_1 g_2 e}{16\pi^2(\hbar c)^2\varkappa_\pi}\,\mathrm{Sp}\,(\gamma_5\gamma_\rho\gamma_\sigma\gamma_\alpha\gamma_\mu)\,U\chi_{\rho\sigma}\frac{\partial A_\mu}{\partial x_\alpha}\left(\frac{\varkappa_\pi^2}{\varkappa^2}+\frac{\varkappa_\tau^2}{\varkappa^2}\right), \tag{25}$$

$$H_2(ps)=\frac{G_2 g_2 e\varkappa}{8\pi^2(\hbar c)^2\varkappa_\tau\varkappa_\pi}\,\mathrm{Sp}\,(\gamma_5\gamma_\rho\gamma_\sigma\gamma_\alpha\gamma_\mu)\,U\chi_{\rho\sigma}\frac{\partial A_\mu}{\partial x_\alpha}\left(\frac{\varkappa_\pi^2}{\varkappa^2}+\frac{\varkappa_\tau^2}{\varkappa^2}\right). \tag{26}$$

Therefore, if we put $G_1=(2\varkappa/\varkappa_\tau)G_2$, obviously $H_1(ps)=H_2(ps)$ holds.

Of course, the identical relations hold for remainders. If we apply the condition $\int(\rho(\varkappa)/\sqrt{\varkappa})\,d\varkappa=0$ for the second term of (24), the identical relations are destroyed in this case as stated in part I.

3) Discussions.

For removal of the descrepancy with regard to the conditions of regulators, as mentioned in Part 1, we make use of the regulating method such as to regulate the both sides of the equality with the same conditions.

Among the two alternatives mentioned in Part 1, the method in which H_1 is regulated after multiplying $\varkappa$ is employed because of its easiness of computation. Truly, identity has the form $H_2=2\varkappa H_1$ in every case and $\varkappa$ appeared in the right hand side of the above relation should not be discriminated from the one to be regulated.

As we have tried to standardize $\varkappa$ for letting the divergent or conditionally convergent terms caused from the singularities of Δ-function separate and drop them off, it is supposed to be natural that if $\varkappa$ of the right hand side of identity were overlooked, the terms which should be removed become uncertain. This is undesirable.

According to the above consideration, identity $[\varkappa^0]=2\varkappa[\varkappa^1]$ gets the form containing even power of $\varkappa$ in both side and $[\varkappa^1]=2\varkappa[\varkappa^0]$ odd power of $\varkappa$.

The terms which do not satisfy the identity but is gauge covariant in the case 1) can be removed by the condition $\int\rho(\varkappa)\,d\varkappa=0$ $(\varkappa=m^2c^2/\hbar^2)$ and the terms do not satisfy the identity in the case 2) can also be dropped off by the condition $\int\sqrt{\varkappa}\rho(\varkappa)\,d\varkappa=0$. This treatment is confirmed by the following consideration: $[\varkappa^0]$ has less power by 1 than $[\varkappa^1]$ with regard to $\varkappa$, therefore, the terms considered to be of the order of $\varkappa^{-n}$ at first sight in H_1 in the case 1) are actually of the order of $\varkappa^{-(n-2)}$ after multiplied by $\varkappa$. This seems to be the reason why the identical relation holds between the term of the order of $\Box^2/\varkappa^2$ in H_2 and the term of the order of $\Box^4/\varkappa^4$ in H_1. In the case 2) such situation does not occur.

Or, if we notice the fact that the differentiation of the wave function is equivalent to the differentiation of $S(x)$, and the differentiation of $S(x)$ has the same dimension as multiplication of $\varkappa$, the procedures mentioned above are easily convinced. Under these considerations, these may be extented to the cases where

$B \neq 0$ (scalar and pseudoscalar mesons). After all, we may conclude as the following: if we admit the following conditions imposed on regulators,

$$\int_{-\infty}^{\infty}\rho(x)\,dx=0,\ \int_{-\infty}^{\infty}x\rho(x)\,dx=0,\ \int_{-\infty}^{\infty}\rho(x)\log|x|\,dx=0,\ \int_{-\infty}^{\infty}x\rho(x)\log|x|\,dx=0,$$

$$\int_{-\infty}^{\infty}\sqrt{x}\rho(x)\,dx=0 \quad (x=m^2c^2/\hbar^2) \tag{27}$$

and regulate H_1 always after multiplying x, then, as long as our cases of transitions are concerned, the identity and gauge-covariance can always be established automatically and consistently.*

§ 6. Decay life time

Up to the present knowledge of the experimental data, the values of coupling constants and the masses of τ^0 and π^0 mesons are not yet known.

Therefore it is impossible to evaluate the decay life times precisely, but we estimate them by assuming that $G_\tau^2/\hbar c \simeq 10^{-1}$, $x_\pi^{\pm}=x_\pi^{\mp}=285m_e$, $x_\tau^{\pm}=600m_e$, in order to study the order of life times in such processes. These choices are due to the experiments of Rochester Butler ($\tau^0\rightarrow\pi^++\pi^-$) and L. Leprince-Ringuet (τ^- yields a star). Further we have supposed that the process $\tau^{\pm}\rightarrow\pi^{\pm}+\tau^0$ may occur. We will reckon the life times about the several examples.**

(*ex.*1) $G_1g_1f_1$ of $s\rightarrow ps+ps$ (Table I)

$$H=\frac{2xG_1g_1f_1}{(2\pi)^2(\hbar c)^2}UVW\int_0^{\infty}\frac{dw}{w}\cos w-\frac{xG_1g_1f_1}{(2\pi)^2(\hbar c)^2}UVW$$

$$+\frac{xG_1g_1f_1}{2(2\pi)^2(\hbar c)^2}\left(2\frac{x_\pi^2}{x^2}-\frac{1}{3}\frac{x_\tau^2}{x^2}\right)UVW+\frac{xG_1g_1f_1}{4(2\pi)^2(\hbar c)^2}UVW\times$$

$$\times\left(\frac{2}{3}\frac{x_\pi^4}{x^4}-\frac{1}{30}\frac{x_\tau^4}{x^4}\right)+\cdots\cdots$$

The first term is dropped off by the condition $\int\rho(x)\,x\log|x|\,dx=0$.***
The life time due to each term is the following:

$$\tau_1^{-1}=T \quad (\textit{from the second term})$$

$$\tau_2^{-1}=\frac{1}{4}\left(2\frac{x_\pi^2}{x^2}-\frac{1}{3}\frac{x_\tau^2}{x^2}\right)^2T$$

---

* As stated in part I, if we employ as the identity $H_1=(2x)^{-1}H_2$, then the further condition $\int\rho(x)/\sqrt{x}\,dx=0$ must be added to let the identity hold in the case 1) however, in this choice, the 2nd and the 4th condition of (27) is unnecessary for the 3rd order transition processes.

** We calculated the decay life times only for the symmetrical theory.

*** As stated in § 5, H is regulated after multiplying x.

$$\tau_3^{-1}=\frac{1}{16}\left(\frac{2}{3}\,\frac{\varkappa_\pi^4}{\varkappa^4}-\frac{1}{30}\,\frac{\varkappa_\tau^4}{\varkappa^4}\right)^2 T.$$

where
$$T=\frac{1}{8\pi^2}\,\frac{G_1^2 g_1^2 f_1^2}{(\hbar c)^3}\,\varkappa_\tau c\,\frac{\varkappa^2(\varkappa_\tau^2-4\varkappa_\pi^2)^{\frac{1}{2}}}{\varkappa_\tau^3}.$$

(*ex.* 2) $G_1 g_1 f_2$ of $s\rightarrow ps+ps$ (Table I)

$$H=-\frac{\varkappa G_1 g_1 f_1}{2(2\pi)^2(\hbar c)^2}\,\delta UVW\left(\frac{\varkappa_\pi^2}{\varkappa^2}-\frac{\varkappa_\tau^2}{\varkappa^2}\right)\int_0^\infty \frac{dw}{w}\cos w$$
$$+\frac{\varkappa G_1 g_1 f_1}{2(2\pi)^2(\hbar c)^2}\,\delta\left(\frac{20}{3}\,\frac{\varkappa_\pi^2}{\varkappa^2}-\frac{23}{3}\,\frac{\varkappa_\tau^2}{\varkappa^2}\right)$$
$$+\frac{\varkappa G_1 g_1 f_1}{12(2\pi)^2(\hbar c)^2}\,\delta\times\left(2\,\frac{\varkappa_\pi^4}{\varkappa^4}+\frac{\varkappa_\tau^4}{\varkappa^4}\right)+\cdots\cdots$$

where
$$\delta=\frac{g_2}{g_1}\,\frac{2\varkappa}{\varkappa_\pi}$$

The first diverging term is dropped off by the condition $\int\rho(\varkappa)\log|\varkappa|\,d\varkappa=0$, and this condition together with the condition $\int\rho(\varkappa)d\varkappa=0$ also remove the 2nd term. The life time due to each term is:

$$\tau_1^{-1}=\frac{1}{64}\,\delta^2\left(\frac{20}{3}\,\frac{\varkappa_\pi^2}{\varkappa^2}-\frac{23}{3}\,\frac{\varkappa_\tau^2}{\varkappa^2}\right)^2 T \quad (\textit{from the second term})$$

$$\tau_2^{-1}=\frac{1}{144}\,\delta^2\left(2\,\frac{\varkappa_\pi^4}{\varkappa^4}+\frac{\varkappa_\tau^4}{\varkappa^4}\right)^2 T$$

The numerical values of these life times are listed in Table IV.

(*ex.* 3) $G_1 g_1 f_1$ of $S\rightarrow S+S$ (Table I, and Table II).

Table IV. (in sec.)

ex. 1	$\tau^+\rightarrow\pi^++\pi^0$	(B) (A)	3.5×10^{-18} 1.56×10^{-22}	0.9×10^{-14} 0.32×10^{-18}	2.3×10^{-11} 0.25×10^{-14}
	$\tau^0\rightarrow\pi^++\pi^-$	(B) (A)	6.5×10^{-18} 2.9×10^{-22}	1.6×10^{-14} 0.7×10^{-18}	4.3×10^{-11} 0.45×10^{-14}
ex. 2	$\tau^+\rightarrow\pi^++\pi^0$	(A) (B)		0.31×10^{-20} 0.5×10^{-18}	1×10^{-17} 1.6×10^{-15}
	$\tau^0\rightarrow\pi^++\pi^-$	(A) (B)		0.56×10^{-20} 0.9×10^{-18}	1.8×10^{-17} 2.9×10^{-15}
ex. 3	$\tau^+\rightarrow\pi^++\pi^0$		0.96×10^{-19}	2.1×10^{-16}	7×10^{-13}
	$\tau^0\rightarrow\pi^++\pi^-$		0.78×10^{-19}	6.0×10^{-16}	0.9×10^{-12}
	$\tau^+\rightarrow\pi^++\tau^0$		4×10^{-17}	4.8×10^{-14}	1.68×10^{-10}

(A) is the one obtained by putting $g_2^2/\hbar c=1/10$ under the relation $g_1=2\varkappa/\varkappa_\pi\, g_2$.

(B) is the one without using the above relation.

Only the numerical value of the life time is listed in Table IV.

The first column can be removed by the regulator conditions $\int \log|\varkappa|\rho(\varkappa)d\varkappa=0$, and $\int \varkappa\rho(\varkappa)\,d\varkappa=0$*. The second column may also be removed by the conditions $\int\rho(\varkappa)\,d\varkappa=0$ and $\int\rho(\varkappa)\log|\varkappa|\,d\varkappa=0$.

(*ex.* 4) $G_1g_1F_1$ and $G_2g_1F_1$ of $v\rightarrow ps+v$ (Table II).

Hamiltonians of these processes are given by (A, 8), (A, 9), (A, 10) and (A, 11). Their life times are the following:

$G_1g_1F_1$ (A) 1.5×10^{-8} sec. from (A, 8), 2.0×10^{-6} sec. from (A, 10),
(B) 1.2×10^{-10} sec. from (A, 8), 1.6×10^{-8} sec. from (A, 10).

$G_2g_1F_1$ (A) 0.4×10^{-10} sec. from (A, 9), 2.8×10^{-8} sec. from (A, 11),
(B) 0.3×10^{-12} sec. from (A, 9), 2.5×10^{-10} sec. from (A, 11).

These lifetimes depend strongly on the masses of τ^-, π^-, and τ^0 mesons Therefore, these values are not reliable since the constants adopted are not actual, but still these values are considered to be too long. These facts mean that this process can not occur actually.

Furthermore, we may, possibly, say that the lifetimes listed in Table IV except (*ex.* 3) are too short to consider to be reasonable values. Therefore, to explain the decay phenomena without discrepancy, it seems to be better to prohibit them. Ex. 3 is the process to make the explanation of Powell and other's experiment possible by assuming the two successive processes $\tau^+\rightarrow\pi^++\tau^0$ and $\tau^0\rightarrow\pi^++\pi^-$.

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Appendix 1.**

In the case of $S\rightarrow S+S$ ($\tau^+\rightarrow\pi^++\pi^0$), the term $G_2g_1f_2$ can be written after the same calculation mentioned at § 2, that is

$$H=\frac{\varkappa G_2g_1f_2}{4\pi^2(\hbar c)^2\varkappa_\tau\varkappa_0}\frac{\partial U}{\partial x_\mu}\frac{\partial V}{\partial x_\mu}W$$

$$-\frac{\varkappa G_2g_1f_2}{8\pi^2(\hbar c)^2\varkappa_\tau\varkappa_0}\frac{\partial U}{\partial x_\mu}\frac{\partial V}{\partial x_\mu}W\int_{-1}^{1}dv\times$$

$$\times\int_1^\infty\frac{du}{u^5}\frac{-(1+v)(1+v-u)\frac{\varkappa_\pi^2}{\varkappa^2}+(1-v)(1-u-v)\frac{\varkappa_0^2}{\varkappa^2}}{1+\frac{(1+v)(1-u)}{2u^2}\frac{\varkappa_\pi^2}{\varkappa^2}+\frac{(1-v)(1-u)}{2u^2}\frac{\varkappa_0^2}{\varkappa^2}-\frac{1-v^2}{4u^2}\frac{\varkappa_\tau^2}{\varkappa^2}}$$

$$-\frac{\varkappa G_2g_1f_2}{8\pi^2(\hbar c)\varkappa_\tau\varkappa_0}\frac{\partial U}{\partial x_\mu}V\frac{\partial W}{\partial x_\mu}\int_{-1}^{1}dv\times$$

---

* Obeying the procedure stated in § 5, these are regulated after multiplying $\varkappa$.

** The following examples are calculated in the neutral meson theory.

$$\times\int_1^\infty\frac{du}{u^5}\frac{(1-v-u)(3+v-2u)\frac{\kappa_\pi^2}{\kappa^2}+(1-v)(1-v-u)\frac{\kappa_0^2}{\kappa^2}-(1-v)(1-v-u)\frac{\kappa_\tau^2}{\kappa^2}}{1+\frac{(1+v)(1-u)}{2u^2}\frac{\kappa_\pi^2}{\kappa^2}+\frac{(1-v)(1-u)}{2u^2}\frac{\kappa_0^2}{\kappa^2}-\frac{1-v^2}{4u^2}\frac{\kappa_\tau^2}{\kappa^2}} \tag{A·1}$$

and this vanishes identically after performing the integration of u and v.

Of course, the first term must be dropped off by the regulating method. $(\int\rho(\kappa)\,d\kappa=0)$

The same result is obtained for the case $G_1g_1f_2$ of $v\rightarrow s+s$ $(\tau^+\rightarrow\pi^++\pi^0)$ except that the factor $(G_2/\kappa_\tau)\partial_\mu U$ should be replaced by $-G_1U_\mu$. Therefore this term also vanishes and these facts seem to give one of the supports to the proof stated at the end of § 3.

Appendix 2.

As another example of the identity of the type $[\kappa^0]=2\kappa[\kappa^1]$, we will show the processes $G_1g_1f_1$ and $G_2g_1f_1$ of $\tau^0\rightarrow\pi^++\pi^-$, when τ-meson is of pseudoscalar type and π-meson of vector type. The Hamiltonian densities are

$$H_{ps}=-\frac{iG_1g_1f_1}{4(2\pi)^2(\hbar c)^2\kappa}T_r(\gamma_5\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta)\,U\frac{\partial V_\mu}{\partial x_\alpha}\frac{\partial W_\nu}{\partial x_\beta}\times$$
$$\times\left\{1-\frac{1}{2}\int_{-1}^1dv\int_1^\infty\frac{du}{u^5}\frac{(1+v)(1-u)\frac{\kappa_2^2}{\kappa^2}+(1-v)(1-u)\frac{\kappa_3^2}{\kappa^2}-\frac{1-v^2}{2}\frac{\kappa_\tau^2}{\kappa^2}}{1+\frac{(1+v)(1-u)}{2u^2}\frac{\kappa_2^2}{\kappa^2}+\frac{(1-v)(1-u)}{2u^2}\frac{\kappa_3^2}{\kappa^2}-\frac{1-v^2}{4u^2}\frac{\kappa_\tau^2}{\kappa^2}}\right. \tag{A·2}$$

$$H_{pv}=-\frac{iG_2g_1f_1}{4(2\pi)^2(\hbar c)^2\kappa}T_r(\gamma_5\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta)U\frac{\partial V_\mu}{\partial x_\alpha}\frac{\partial W_\nu}{\partial x_\beta}\times$$
$$\times\left\{\frac{2}{3}-\int_1^\infty\frac{du}{u^6}\frac{(2-u)(1-u)\frac{\kappa_2^2}{\kappa^2}}{1+\frac{1-u}{u^2}\frac{\kappa_2^2}{\kappa^2}}-\int_1^\infty\frac{du}{u^6}\frac{(2-u)(1-u)\frac{\kappa_3^2}{\kappa^2}}{1+\frac{1-u}{u^2}\frac{\kappa_3^2}{\kappa^2}}+\frac{1}{2}\int_{-1}^1dv\int_1^\infty\frac{du}{u^6}\right.$$
$$\left.\times\frac{(1-u)(2-3u-v-uv)\frac{\kappa_3^2}{\kappa^2}+(1-u)(2-3u+v+uv)\frac{\kappa_3^2}{\kappa^2}+(2u-1-2v^2)\frac{\kappa_\tau^2}{\kappa^2}}{1+\frac{(1+v)(1-u)}{2u^2}\frac{\kappa_2^2}{\kappa^2}+\frac{(1-v)(1-u)}{2u^2}\frac{\kappa_3^2}{\kappa^2}-\frac{1-v^2}{4u^2}\frac{\kappa_\tau^2}{\kappa^2}}\right\} \tag{A·3}$$

Obviously, identical relation does not hold between the first terms. The second terms become respectively

$$H_{ps}=-\frac{iG_1g_1f_1}{48(2\pi)^2\kappa}T_r(\gamma_5\gamma_\alpha\gamma_\beta\gamma_\mu\gamma_\nu)U\frac{\partial V_\mu}{\partial x_\alpha}\frac{\partial W_\nu}{\partial x_\beta}\left(\frac{\kappa_2^2}{\kappa^2}+\frac{\kappa_3^2}{\kappa^2}+\frac{\kappa_\tau^2}{\kappa^2}\right) \tag{A·4}$$

$$H_{pv}=-\frac{iG_2 g_1 f_1}{24(2\pi)^2\varkappa_\tau}T_r(\gamma_5\gamma_\alpha\gamma_\beta\gamma_\mu\gamma_\nu)U\frac{\partial V_\mu}{\partial x_\alpha}\frac{\partial W_\nu}{\partial x_\beta}\left(\frac{\varkappa_2^2}{\varkappa^2}+\frac{\varkappa_3^2}{\varkappa^2}+\frac{\varkappa_\tau^2}{\varkappa^2}\right) \qquad (A\cdot 5)$$

Therefore, if we equate G_1 to $(2\varkappa/\varkappa_\tau)G_2$, $H_{ps}=H_{pv}$. This equality holds for the remainders. If $\varkappa_2=0$, $\varkappa_3=0$, these turn to the case of the γ-decay of pseudoscalar neutretto.

Appendix 3.

As an example of the case where the vector meson is concerned, $G_1 g_1 F_1$ and $G_2 g_1 F_1$ of $\tau^\pm\rightarrow\pi^\pm+\tau^0$ when τ is of vector and π of pseudoscalar are picked up.

$$H_v=\frac{iG_1 g_1 F_1}{4(2\pi)^2(\hbar c)^2\varkappa}T_r(\gamma_5\gamma_\alpha\gamma_\beta\gamma_\mu\gamma_\lambda)U_\mu\frac{\partial V}{\partial x_\alpha}\frac{\partial W_\lambda}{\partial x_\beta}\times$$

$$\times\left\{1-\frac{1}{2}\int_{-1}^{1}dv\int_1^\infty\frac{du}{u^5}\frac{(1+u)(1-u)\frac{\varkappa_2^2}{\varkappa^2}+(1-u)(1-u)\frac{\varkappa_3^2}{\varkappa^2}-\frac{1-v^2}{2}\frac{\varkappa_\tau^2}{\varkappa^2}}{1+\frac{(1+v)(1-u)}{2u^2}\frac{\varkappa_2^2}{\varkappa^2}+\frac{(1-v)(1-u)}{2u^2}\frac{\varkappa_3^2}{\varkappa^2}-\frac{1-v^2}{4u^2}\frac{\varkappa_\tau^2}{\varkappa^2}}\right\}$$

$$(A\cdot 6)$$

$$H_t=\frac{iG_2 g_1 f_1}{8(2\pi)^2(\hbar c)^2\varkappa_\tau}T_r(\gamma_5\gamma_\alpha\gamma_\beta\gamma_\mu\gamma_\lambda)U_\mu\frac{\partial V}{\partial x_\alpha}\frac{\partial W_\lambda}{\partial x_\beta}\left\{2\int_1^\infty\frac{dw}{w}\cos w-\frac{2}{3}\right.$$

$$-\int_1^\infty\frac{du}{u^6}\frac{(2-u)(2-3u)\frac{\varkappa_2^2}{\varkappa^2}}{1+\frac{1-u}{u^2}\frac{\varkappa_2^2}{\varkappa^2}}-\int_1^\infty\frac{du}{u^6}\frac{(2-u)(2-3u)\frac{\varkappa_3^2}{\varkappa^2}}{1+\frac{1-u}{u^2}\frac{\varkappa_3^2}{\varkappa^2}}+\int_{-1}^{1}dv\int_1^\infty\frac{du}{u^6}\times$$

$$\left.\times\frac{(1-u)(2-3u-v)\frac{\varkappa_2^2}{\varkappa^2}+(1-u)(2-3u+v)\frac{\varkappa_3^2}{\varkappa^2}-(1+2v^2-2u-uv^2)\frac{\varkappa_\tau^2}{\varkappa^2}}{1+\frac{(1+v)(1-u)}{2u^2}\frac{\varkappa_2^2}{\varkappa^2}+\frac{(1-v)(1-u)}{2u^2}\frac{\varkappa_3^2}{\varkappa^2}-\frac{1-v^2}{4u^2}\frac{\varkappa_\tau^2}{\varkappa^2}}\right\}$$

$$(A\cdot 7)$$

The first term of (A·7) can be removed by the condition $\int\rho(\varkappa)\log|\varkappa|d\varkappa=0$.

The second term of (A·7) and the first term of (A·6) may be dropped off by the condition $\int\rho(\varkappa)\log|\varkappa|d\varkappa=0$ and $\int\rho(\varkappa)d\varkappa=0$.

The second term of (A·6) is

$$-\frac{iG_1 g_1 F_1}{48(\hbar c)^2(2\pi)^2\varkappa}T_r(\gamma_5\gamma_\alpha\gamma_\beta\gamma_\mu\gamma_\lambda)U_\mu\frac{\partial V}{\partial x_\alpha}\frac{\partial W_\lambda}{\partial x_\beta}\left(\frac{\varkappa_2^2}{\varkappa^2}+\frac{\varkappa_3^2}{\varkappa^2}+\frac{\varkappa_\tau^2}{\varkappa^2}\right) \qquad (A\cdot 8)$$

The third term of (A·7) can be written

$$-\frac{iG_2 g_1 F_1}{16(2\pi)^2(\hbar c)^2\varkappa_\tau}T_r(\gamma_5\gamma_\alpha\gamma_\beta\gamma_\mu\gamma_\lambda)U_\mu\frac{\partial V}{\partial x_\alpha}\frac{\partial W_\lambda}{\partial x_\beta}\left(\frac{\varkappa_2^2}{\varkappa^2}+\frac{\varkappa_3^2}{\varkappa^2}+\frac{\varkappa_\tau^2}{\varkappa^2}\right). \qquad (A\cdot 9)$$

The third term of (A·6) reads

$$-\frac{iG_1 g_1 F_1}{8\cdot45(2\pi)^2(\hbar c)^2\varkappa} T_r(\gamma_5\gamma_\alpha\gamma_\beta\gamma_\mu\gamma_\lambda)U_\mu \frac{\partial V}{\partial x_\alpha}\frac{\partial W_\lambda}{\partial x_\beta}$$
$$\times\left(\frac{\varkappa_2^4}{\varkappa^4}+\frac{\varkappa_3^4}{\varkappa^4}+\frac{\varkappa_\tau^4}{\varkappa^4}+\frac{\varkappa_2^2\varkappa_3^2}{\varkappa^4}+\frac{\varkappa_2^2\varkappa_\tau^2}{\varkappa^4}+\frac{\varkappa_3^2\varkappa_\tau^2}{\varkappa^4}\right). \tag{A·10}$$

The fourth term of (A·7) becomes

$$-\frac{iG_2 g_1 F_1}{8\cdot18(2\pi)^2(\hbar c)^2\varkappa_\tau} T_r(\gamma_5\gamma_\alpha\gamma_\beta\gamma_\mu\gamma_\lambda)U_\mu \frac{\partial V}{\partial x_\alpha}\frac{\partial W_\lambda}{\partial x_\beta}$$
$$\times\left(\frac{\varkappa_2^4}{\varkappa^4}+\frac{\varkappa_3^4}{\varkappa^4}+\frac{\varkappa_\tau^4}{\varkappa^4}+\frac{\varkappa_2^2\varkappa_3^2}{\varkappa^4}+\frac{\varkappa_2^2\varkappa_\tau^2}{\varkappa^4}+\frac{\varkappa_3^2\varkappa_\tau^2}{\varkappa^4}\right). \tag{A·11}$$

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Spinors in Five Dimensions

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Introduction

Several authors have recently made many interesting considerations about five-dimensional spaces, so we have studied the spinor transformations in five dimensions, connecting van der Waerden's and Brauer-Weyl's standpoints.[1]

§ 1. Spinor transformation

Let us consider the linear transformations of five variables $x_1, \ldots\ldots, x_5$ which leave

$$x_1^2+x_2^2+x_3^2-x_4^2+x_5^2$$

invariant, and call it as de Sitter transformation. Brauer and Weyl showed that, if we have five matrices satisfying

$$P_i P_j + P_j P_i = 2g_{ij} E$$

where

$$g_{11}=g_{22}=g_{33}=-g_{44}=g_{55}=1, \quad g_{ij}=0 \quad (i \neq j)$$

$$E: \quad \text{unit matrix}$$

there exists to any de Sitter transformation (a_{ij}) the spinor transformation S with the correlation

$$S P_i S^{-1} = \sum_j a_{ji} P_j,$$

and they gave a concrete representation of P_k. But we adopt another representation of P_k in order to connect Brauer-Weyl's and van der Waerden's standpoints and to obtain van der Waerden's form of spinors when the transformation reduces to a Lorentz transformation, namely

$$P_1=\left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right), \quad P_2=\left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right), \quad P_3=\left(\begin{array}{cc|cc} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ \hline 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{array}\right),$$

$$P_4=\left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right), \quad P_5=\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right).$$

We can easily obtain the spinor transformations S_{ij}'s corresponding to the ij-plane transformations. The most general spinor transformation, which we seek, is one which is produced from all these S_{ij}'s by multiplying them in all possible manners. Therefore we seek a general form of S which comprises all S_{ij}'s as its special cases and leaves its form invariant to any multiplication among themselves.

In this way we have obtained

$$S=\left(\begin{array}{cc|cc} \alpha & \beta & \lambda & \mu \\ \gamma & \delta & \nu & \sigma \\ \hline \bar{\sigma} & -\bar{\nu} & \bar{\delta} & -\bar{\gamma} \\ -\bar{\mu} & \bar{\lambda} & -\bar{\beta} & \bar{\alpha} \end{array}\right).$$

In the same way, we have obtained, as the form of the inverse transformation of S,

$$T=\left(\begin{array}{cc|cc} \delta & -\beta & -\bar{\lambda} & -\bar{\nu} \\ -\gamma & \alpha & -\bar{\mu} & -\bar{\sigma} \\ \hline -\sigma & \mu & \bar{\alpha} & \bar{\gamma} \\ \nu & -\lambda & \bar{\beta} & \bar{\delta} \end{array}\right).$$

In order that T is exactly the inverse of S, it is necessary that

$$ST=TS=E.$$

Thus we have the following conditions between the elements of the matrix S,

$$\left.\begin{array}{l} \alpha\delta-\beta\gamma-\lambda\sigma+\mu\nu=1 \\ -\alpha\bar{\nu}-\beta\bar{\sigma}+\bar{\gamma}\lambda+\bar{\delta}\mu=0 \\ \lambda\bar{\alpha}+\mu\bar{\beta}=\text{real} \\ \gamma\bar{\nu}+\delta\bar{\sigma}=\text{real} \end{array}\right\},$$

or

$$\left.\begin{array}{l} \alpha\delta-\beta\gamma-\bar{\lambda}\bar{\sigma}+\bar{\mu}\bar{\nu}=1 \\ -\bar{\alpha}\bar{\nu}-\beta\sigma+\bar{\gamma}\bar{\lambda}+\delta\mu=0 \\ \alpha\sigma-\gamma\mu=\text{real} \\ \delta\lambda-\beta\nu=\text{real} \end{array}\right\}.$$

These conditions confine the six degrees of freedom, so that S has $16-6=10$ degrees of freedom. Indeed the de Sitter group is a 10 parameter group.

We can easily verify that

$$\det\ S=1.$$

Forming the matrix

$$X=x_1P_1+x_2P_2+x_3P_3+x_4P_4+x_5P_5$$

$$=\begin{pmatrix} x_5 & 0 & x_2+x_4 & x_1+ix_3 \\ 0 & x_5 & x_1-ix_3 & -x_2+x_4 \\ x_2-x_4 & x_1+ix_3 & -x_5 & 0 \\ x_1-ix_3 & -x_2-x_4 & 0 & -x_5 \end{pmatrix},$$

its determinant becomes

$$\det\ X=(x_1^2+x_2^2+x_3^2-x_4^2+x_5^2)^2.$$

We dan easily calculate the transformed matrix Y from X by S, and then we see that it has the same form as X,

$$Y=SXS^{-1}=\begin{pmatrix} y_5 & 0 & y_2+y_4 & y_1+iy_3 \\ 0 & y_5 & y_1-iy_3 & -y_2+y_4 \\ y_2-y_4 & y_1+iy_3 & -y_5 & 0 \\ y_1-iy_3 & -y_2-y_4 & 0 & -y_5 \end{pmatrix},$$

and that the transformation $x\rightarrow y$ is a real one and the coefficient corresponding to $x_4\rightarrow y_4$ is positive. From

$$\det\ X=\det\ Y$$

it holds,

$$y_1^2+y_2^2+y_3^2-y_4^2+y_5^2=\pm(x_1^2+x_2^2+x_3^2-x_4^2+x_5^2).$$

According to the inertia theorem

$$y_1^2+y_2^2+y_3^2-y_4^2+y_5^2=x_1^2+x_2^2+x_3^2-x_4^2+x_5^2$$

Thus we have shown that $x\rightarrow y$ is a de Sitter transformation and S is the corresponding spinor transformation.

Fixing the fifth coordinate x_5, we have a Lorentz transformation between x_1, x_2, x_3, x_4 which leaves

$$x_1^2+x_2^2+x_3^2-x_4^2$$

invariant. From

$$S_LP_5S_L^{-1}=P_5,$$

We obtain the form of S_L as

$$S_L=\begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix},$$

where clearly

$$S_1=\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad S_2=\begin{pmatrix} \bar{\delta} & -\bar{\gamma} \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix},$$

and

$$\lambda=\mu=\nu=\sigma=0.$$

The conditions reduce to only one, as well known,

$$\alpha\delta-\beta\gamma=1,$$

therefore S_2 can be expressed by S_1,

$$S_2=\overline{(S_1^{-1})}',$$

where the bar indicates the complex conjugate and dash the transposed matrix.

Fixing further the forth coordinate x_4 also, we have a three dimensional space rotation between x_1, x_2, x_3 which leaves

$$x_1^2+x_2^2+x_3^2$$

invariant. From

$$S_R P_4 R_R^{-1}=P_4$$

we see

$$S_1=S_2,$$

so we have the general form of S_R as

$$S_R=\begin{pmatrix} S_1 & 0 \\ 0 & S_1 \end{pmatrix},$$

where

$$\delta=\bar{\alpha}, \quad \gamma=-\bar{\beta}.$$

Thus we obtain

$$S_1=\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}.$$

The condition reduces to

$$\alpha\bar{\alpha}+\beta\beta=1.$$

§ 2. Spinor of the first rank.

Let us call a covariant spinor the quantity constituted of the four components $\Psi_1, \Psi_2, \Psi_3, \Psi_4$ transforming according to the above obtained spinor transformation S. On the other hand let us call a contravariant spinor the quantity consitituted of the four components Φ^1, Φ^2, Φ^3, Φ^4 transforming contragradiently to S, namely according to

$$(S^{-1})'=\left(\begin{array}{cc|cc} \delta & -\gamma & -\sigma & \gamma \\ -\beta & \alpha & \mu & -\lambda \\ \hline -\bar{\lambda} & -\bar{\mu} & \bar{\alpha} & \bar{\beta} \\ -\bar{\nu} & -\bar{\sigma} & \bar{\gamma} & \bar{\delta} \end{array}\right)$$

Of course, their inner product

$$(\Phi\Psi)=\Phi^1\Psi_1+\Phi^2\Psi_2+\Phi^3\Psi_3+\Phi^4\Psi_4$$

is invariant.

In order to obtain relation between covariant and contravariant spinors, we observe that if we have a matrix C which satisfies

$$(S^{-1})'=CSC^{-1},$$

then $C\Psi$, formed from a covariant Ψ, becomes a contravariant spinor. From the given form of S and $(S^{-1})'$ we obtain for C

$$C=(C_{ij})=\left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right).$$

Then we can define the operation of raising and lowering suffixes, as follows

$$\Psi^i=\sum_k C^{ik}\Psi_k,\quad \Psi_i=\sum_k C_{ik}\Psi^k$$

We can also write these operations as follows

$$(\Psi^1,\Psi^2,\Psi^3,\Psi^4)=(\Psi_2,-\Psi_1,-\Psi_4,\Psi_3)$$

$$(\Psi_1,\Psi_2,\Psi_3,\Psi_4)=(-\Psi^2,\Psi^1,\Psi^4,-\Psi^3)$$

As above shown, confining the transformation to a proper Lorentz transformation by fixing the fifth coordinate x_5, S splits into two parts S_1 and S_2. Therefore (Ψ_1,Ψ_2) undergoes S_1, and (Ψ_3,Ψ_4) undergoes S_2. As for the contravariant spinors, $(S^{-1})'$ becomes as easily shown,

$$(S^{-1})'=\begin{pmatrix} (S_1^{-1})' & 0 \\ 0 & (S_2^{-1})' \end{pmatrix}=\left(\begin{array}{cc|cc} \delta & -\gamma & 0 & 0 \\ -\beta & \alpha & 0 & 0 \\ \hline 0 & 0 & \bar{\alpha} & \beta \\ 0 & 0 & \bar{\gamma} & \bar{\delta} \end{array}\right)=\begin{pmatrix} \bar{S}_2 & 0 \\ 0 & \bar{S}_1 \end{pmatrix},$$

so that (Ψ^1,Ψ^2) transforms in the same way as $(\bar{\Psi}_3,\bar{\Psi}_4)$, and (Ψ^3,Ψ^4) as $(\Psi_1,\bar{\Psi}_2)$.

The qnantity (ψ_1,ψ_2) which transforms according to

$$S_1=\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \alpha\delta-\beta\gamma=1,$$

and the quantity $(\psi_{\dot{1}}, \psi_{\dot{2}})$ which transforms according to

$$\bar{S}_1=\begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\gamma} & \bar{\delta} \end{pmatrix},$$

are the spinors introduced by van der Waerden. Thus confining the transformation to a Lorentz transformations, our spinors become to the van der Waerden's, namely

$$(\Psi_1, \Psi_2, \Psi_3, \Psi_4) \rightarrow (\psi_1, \psi_2, \phi^{\dot{1}}, \phi^{\dot{2}})$$

$$(\Psi^1, \Psi^2, \Psi^3, \Psi^4) \rightarrow (\psi^1, \psi^2, \phi_{\dot{1}}, \phi_{\dot{2}})$$

Further confining the transformation to a three dimensional space rotation by fixing the fourth axis too S_1 becomes equal to S_2 and

$$\bar{S}_1=(S_1^{-1})',$$

so $(\psi^{\dot{1}}, \psi^{\dot{2}})$ transforms in the same manner as (ψ_1, ψ_2), and $(\psi_{\dot{1}}, \psi_{\dot{2}})$ as (ψ^1, ψ^2).

But these are all proper Lorentz transformations whose determinate are $+1$. In order to consider improper Lorentz transformation L^-, we make a proper de Sitter transformation by combining L^- with $x_5'=-x_5$. Since all L^- can be obtained as product of an arbitrarily fixed L^- and a suitable proper tansformation L^+, whose determinant is $+1$, we choose as fixed L^-, $x_4'=-x_4$, $x_i'=x_i (i=1, 2, 3)$.

The corresponding spinor transformation is

$$\left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right),$$

namely

$$(\Psi_1, \Psi_2, \Psi_3, \Psi_4) \rightarrow (\Psi_3, \Psi_4, \Psi_1, \Psi_2)$$

or

$$\left.\begin{array}{l} (\psi_1, \psi_2) \rightarrow (\phi^{\dot{1}}, \phi^{\dot{2}}) \\ (\psi^{\dot{1}}, \psi^{\dot{2}}) \rightarrow (\phi_1, \phi_2) \end{array}\right\}.$$

Thus spinors corresponding to an improper Lorentz transformation is transformed by the above one combined, with an ordinary one.

§ 3. Spinors of the second and higher rank

Spinors of higher rank can be defined in the same way as the case of tensor analysis, namely as a quantity transforming according to so-called product transformations. For example, mixed spinor Ψ_i^l of the second rank transforms as a product $\Psi_i \Phi^l$ of two spinors.

But higher rank spinors are in general not irreducible. In the four dimensional case all irreducible spinors can be obtained by symmetrisation, but in the five dimensions this is not the case.

In the latter case, although symmtry spinors are always irreducible, all irreducible spinors can not necessarily be expressed by symmetry spinors only. Thus it is a very complicated matter to choose irreducible parts from general higher rank spinors, but in the case of lower ranks it can be performed exactly. For example, let us treat the case of second rank.

The second rank spinors are composed of symmetrical and antisymmetrical parts. The symmetrical spinor can be shown to be irreducible. As easily shown, confining the transformation to proper Lorentz transformation, it reduces to symmetrical spinors of van der Wearden as follows. $\{\Psi_{11}, \Psi_{12}, \Psi_{22}, \Psi_{33}, \Psi_{34}, \Psi_{44};$ $\Psi_{13}, \Psi_{14}, \Psi_{23}, \Psi_{24}\} \to \{(\varphi_{11}, \varphi_{12}, \varphi_{22}), (\varphi^{\dot{1}\dot{1}}, \varphi^{\dot{1}\dot{2}}, \varphi^{\dot{2}\dot{2}}), (\varphi_1^{\dot{1}}, \varphi_1^{\dot{2}}, \varphi_2^{\dot{1}}, \varphi_2^{\dot{2}})\}$. These three parts transform as irreducible transformations under proper Lorentz transformations.

As the transformations of our spinors can be connected by Weyl's unitary trick with the corresponding rotation of five dimmensional Euclidean space, the irreducibility of our spinors is assured by the irreducibility of the corresponding Euclidean spinors. Also in the Euclidean case, fixing the fifth axis, the spinor transformation reduces to the form

$$T^0 = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & B_2 & 0 \\ 0 & 0 & C_3 \end{pmatrix}$$

A_1, B_2, C_3 being all irreducible, the matrices commutative with all T^0 must have the form

$$U = \begin{pmatrix} \alpha E_1 & 0 & 0 \\ 0 & \beta E_2 & 0 \\ 0 & 0 & \gamma E_3 \end{pmatrix}$$

Setting the fifth axis free, nondiagonal parts of T appear. Therefore in order to be commutative with all these T, α, β, γ must be all equal and U must be a constant multiple of the unit matrix.

As rotation group is bounded, T is either completely reducible or irreducible. Therefore by the Schur's lemma T must be irreducible, so that our symmetrical spinors of the second rank are also irreducible.

Similarly we can verify that symmetrical spinors of any rank are also irreducible.

About the antisymmetrical spinors of the second rank we can perform the same analysis. They are not irreducible and splits into two irreducible parts, the one is

$$(\Psi_{13}, \Psi_{14}, \Psi_{23}, \Psi_{24}, \Psi_{12}+\Psi_{34})$$

which equivalent to a five dimensional vector, and the other is

$$\Psi_{12}-\Psi_{34}$$

which is a scalar. The former reduces in the case of Lorentz transforemation to the two parts

$$(\Psi_{13}, \Psi_{14}, \Psi_{23}, \Psi_{24}) \rightarrow (\psi_1^{\dot 1}, \psi_1^{\dot 2}, \psi_2^{\dot 1}, \psi_2^{\dot 2})$$

and

$$\Psi_{12}+\Psi_{34} \rightarrow \text{scalar.}$$

In order to show these circumstances more clearly, let us write down the transformation explicitly.

$$\begin{pmatrix} \Psi'_{13} \\ \Psi'_{14} \\ \Psi'_{23} \\ \Psi'_{24} \\ \Psi'_{12}+\Psi'_{34} \\ \Psi'_{12}-\Psi'_{34} \end{pmatrix} = \left(\begin{array}{cccc|c|c} \alpha\bar\delta-\lambda\bar\sigma & -\alpha\bar\gamma-\mu\bar\sigma & \beta\bar\delta+\lambda\bar\nu & -\beta\bar\gamma+\mu\bar\nu & -\alpha\bar\nu-\beta\bar\sigma & 0 \\ -\alpha\bar\beta+\lambda\bar\mu & \alpha\bar\alpha+\mu\bar\mu & -\beta\bar\beta-\lambda\bar\lambda & \beta\bar\alpha-\mu\bar\lambda & \alpha\bar\lambda+\beta\bar\mu & 0 \\ \gamma\bar\delta-\nu\bar\sigma & -\gamma\bar\gamma-\sigma\bar\sigma & \delta\bar\delta+\nu\bar\nu & -\delta\bar\gamma+\sigma\bar\nu & -\gamma\bar\nu-\delta\bar\sigma & 0 \\ \gamma\bar\beta+\nu\bar\mu & \gamma\bar\alpha+\sigma\bar\mu & -\delta\bar\beta-\nu\bar\lambda & \delta\bar\alpha-\sigma\bar\lambda & \gamma\bar\lambda+\delta\bar\mu & 0 \\ \cline{1-4} \alpha\nu-\lambda\gamma & \alpha\sigma-\mu\gamma & \beta\nu-\lambda\delta & \multicolumn{1}{c}{\beta\sigma-\mu\delta} & \alpha\delta-\beta\gamma+\lambda\sigma-\mu\nu & 0 \\ \cline{1-5} 0 & 0 & 0 & \multicolumn{1}{c}{0} & \multicolumn{1}{c|}{0} & 1 \end{array}\right) \begin{pmatrix} \Psi_{13} \\ \Psi_{14} \\ \Psi_{23} \\ \Psi_{24} \\ \Psi_{12}+\Psi_{34} \\ \Psi_{12}-\Psi_{34} \end{pmatrix}$$

§ 4. An application to wave equations of elementary particles

Bhabha has recently made an interesting consideration about relativistic wave equations from a five dimensional standpoint. Though his discussion is made using the theory of group representation, but with our five dimensional spinor analysis we can investigate not only the form of wave equation but also the details of wave functions. Bhabha's equation is

$$\{p_k \alpha^k + \varkappa\} \psi = 0,$$

where p_k is differrential operator by x_k, $\varkappa$ is invariant mass term, α^k is written with an irreducible representation of five dimensional rotation group I^{mn} as follows

$$\alpha^k = I^{kt},$$

ψ is reduced spinor from the irreducible five dimensional spinor corresponding to the representation I^{mn} by confining to the four dimensional transformation. We can obtain these I^{mn} from our spinor transformation

$$I^{mn} = \left(\frac{\partial S_{mn}(\theta)}{\partial \theta}\right)_{\theta=0},$$

therefore

$$a^k=\left(\frac{\partial S_{k5}(\theta)}{\partial\theta}\right)_{\theta=0}.$$

For example, the wave function of meson is second rank spinor, namely the vector meson field is symmetrical spinor and the scalar meson field is irreducible part of antisymmetical spinor.

In the case of vetor meson field, we put for symmetrical spinors of the second rank

$$(\xi_9, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6, \xi_7, \xi_8, \xi_9, \xi_{10})=(\Psi_{11}, \Psi_{12}, \Psi_{22}, \Psi_{33}, \Psi_{34}, \Psi_{44};\ \Psi_{13}, \Psi_{14}, \Psi_{23}, \Psi_{24}).$$

The transformation $S_{i5}(\theta)$ of these quantity corresponding to the $i5$-plane Lorentz transformation by angle θ becomes as follows:

$$S_{15}(\theta)=\left(\begin{array}{cccccc|cccc}
c^2 & 0 & 0 & 0 & 0 & s^2 & 0 & -\sqrt{2}sc & 0 & 0 \\
0 & c^2 & 0 & 0 & s^2 & 0 & -sc & 0 & 0 & -sc \\
0 & 0 & c^2 & s^2 & 0 & 0 & 0 & 0 & -\sqrt{2}sc & 0 \\
0 & 0 & s^2 & c^2 & 0 & 0 & 0 & 0 & \sqrt{2}sc & 0 \\
0 & s^2 & 0 & 0 & c^2 & 0 & sc & 0 & 0 & sc \\
s^2 & 0 & 0 & 0 & 0 & c^2 & 0 & \sqrt{2}sc & 0 & 0 \\
\hline
0 & sc & 0 & 0 & -sc & 0 & c^2 & 0 & 0 & -s^2 \\
\sqrt{2}sc & 0 & 0 & 0 & 0 & -\sqrt{2}sc & 0 & c^2-s^2 & 0 & 0 \\
0 & 0 & \sqrt{2}sc & -\sqrt{sc} & 0 & 0 & 0 & 0 & c^2-s^2 & 0 \\
0 & sc & 0 & 0 & -sc & 0 & -s^2 & 0 & 0 & c^2
\end{array}\right),$$

$$S_{25}(\theta)=\left(\begin{array}{cccccc|cccc}
c^2 & 0 & 0 & s^2 & 0 & 0 & -\sqrt{2}sc & 0 & 0 & 0 \\
0 & c^2 & 0 & 0 & -s^2 & 0 & 0 & sc & -sc & 0 \\
0 & 0 & c^2 & 0 & 0 & s^2 & 0 & 0 & 0 & \sqrt{2}sc \\
s^2 & 0 & 0 & c^2 & 0 & 0 & \sqrt{2}sc & 0 & 0 & 0 \\
0 & -s^2 & 0 & 0 & c^2 & 0 & 0 & sc & -sc & 0 \\
0 & 0 & s^2 & 0 & 0 & c^2 & 0 & 0 & 0 & -\sqrt{2}sc \\
\hline
\sqrt{2}sc & 0 & 0 & -\sqrt{2}sc & 0 & 0 & c^2-s^2 & 0 & 0 & 0 \\
0 & -sc & 0 & 0 & -sc & 0 & 0 & c^2 & s^2 & 0 \\
0 & sc & 0 & 0 & sc & 0 & 0 & s^2 & c^2 & 0 \\
0 & 0 & -\sqrt{2}sc & 0 & 0 & \sqrt{2}sc & 0 & 0 & 0 & c^2-s^2
\end{array}\right),$$

$$S_{35}(\theta)=\left(\begin{array}{cccccc|cccc}
c^2 & 0 & 0 & 0 & 0 & -s^2 & 0 & i\sqrt{2}sc & 0 & 0 \\
0 & c^2 & 0 & 0 & s^2 & 0 & isc & 0 & 0 & -isc \\
0 & 0 & c^2 & -s^2 & 0 & 0 & 0 & 0 & i\sqrt{2}sc & 0 \\
0 & 0 & -s^2 & c^2 & 0 & 0 & 0 & 0 & i\sqrt{2}sc & 0 \\
0 & s^2 & 0 & 0 & c^2 & 0 & -isc & 0 & 0 & isc \\
-s^2 & 0 & 0 & 0 & 0 & c^2 & 0 & i\sqrt{2}sc & 0 & 0 \\
\hline
0 & isc & 0 & 0 & -isc & 0 & c^2 & 0 & 0 & s^2 \\
i\sqrt{2}sc & 0 & 0 & 0 & 0 & i\sqrt{2}sc & 0 & c^2-s^2 & 0 & 0 \\
0 & 0 & i\sqrt{2}sc & i\sqrt{2}sc & 0 & 0 & 0 & 0 & c^2-s^2 & 0 \\
0 & -isc & 0 & 0 & isc & 0 & s^2 & 0 & 0 & c^2
\end{array}\right),$$

$$S_{45}(\theta)=\left(\begin{array}{cccccc|cccc}
c'^2 & 0 & 0 & s'^2 & 0 & 0 & -\sqrt{2}s'c' & 0 & 0 & 0 \\
0 & c'^2 & 0 & 0 & s'^2 & 0 & 0 & -s'c' & -s'c' & 0 \\
0 & 0 & c'^2 & 0 & 0 & s'^2 & 0 & 0 & 0 & -\sqrt{2}s'c' \\
s'^2 & 0 & 0 & c'^2 & 0 & 0 & -\sqrt{2}s'c' & 0 & 0 & 0 \\
0 & s'^2 & 0 & 0 & c'^2 & 0 & 0 & -s'c' & -s'c' & 0 \\
0 & 0 & s'^2 & 0 & 0 & c'^2 & 0 & 0 & 0 & -\sqrt{2}s'c' \\
\hline
-\sqrt{2}s'c' & 0 & 0 & -\sqrt{2}s'c' & 0 & 0 & c'^2+s'^2 & 0 & 0 & 0 \\
0 & -s'c' & 0 & 0 & -s'c' & 0 & 0 & c'^2 & s'^2 & 0 \\
0 & -s'c' & 0 & 0 & -s'c' & 0 & 0 & s'^2 & c'^2 & 0 \\
0 & 0 & -\sqrt{2}s'c' & 0 & 0 & -\sqrt{2}s'c' & 0 & 0 & 0 & c'^2+s'^2
\end{array}\right).$$

where we used the following abbreviation

$$s=\sin\frac{\theta}{2},\quad c=\cos\frac{\theta}{2},\quad s'=\sinh\frac{\theta}{2},\quad c'=\cosh\frac{\theta}{2}.$$

So we have a^k,

$$a^1=\frac{i}{2}\left(\begin{array}{c|c}
0 & \begin{matrix} 0 & -\sqrt{2} & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & -\sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 1 & 0 & 0 & 1 \\ 0 & \sqrt{2} & 0 & 0 \end{matrix} \\
\hline
\begin{matrix} 0 & 1 & 0 & 0 & -1 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 & -\sqrt{2} \\ 0 & 0 & \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \end{matrix} & 0
\end{array}\right),$$

$$\alpha^2 = \frac{i}{2} \left(\begin{array}{c|c} 0 & \begin{matrix} -\sqrt{2} & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & \sqrt{2} \\ \sqrt{2} & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -\sqrt{2} \end{matrix} \\ \hline \begin{matrix} \sqrt{2} & 0 & 0 & -\sqrt{2} & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\sqrt{2} & 0 & 0 & \sqrt{2} \end{matrix} & 0 \end{array} \right),$$

$$\alpha^3 = \frac{-1}{2} \left(\begin{array}{c|c} 0 & \begin{matrix} 0 & -\sqrt{2} & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -\sqrt{2} & 0 & 0 \end{matrix} \\ \hline \begin{matrix} 0 & 1 & 0 & 0 & -1 & 0 \\ -\sqrt{2} & 0 & 0 & 0 & 0 & -\sqrt{2} \\ 0 & 0 & \sqrt{2} & \sqrt{2} & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \end{matrix} & 0 \end{array} \right)$$

$$\alpha^4 = \frac{i}{2} \left(\begin{array}{c|c} 0 & \begin{matrix} -\sqrt{2} & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -\sqrt{2} \\ -\sqrt{2} & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -\sqrt{2} \end{matrix} \\ \hline \begin{matrix} -\sqrt{2} & 0 & 0 & -\sqrt{2} & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -\sqrt{2} & 0 & 0 & -\sqrt{2} \end{matrix} & 0 \end{array} \right).$$

If we put

$$\xi_1' = \frac{1}{\sqrt{2}} (\xi_1 - \xi_3 + \xi_4 - \xi_6)$$

$$\xi_2' = -i (\xi_2 + \xi_5)$$

$$\xi_3' = \frac{1}{\sqrt{2}} (\xi_1 + \xi_3 + \xi_4 + \xi_6)$$

$$\xi_4' = \frac{1}{\sqrt{2}} (\xi_1 - \xi_3 - \xi_4 + \xi_6)$$

$$\xi_5'=\xi_2-\xi_5 \qquad \xi_8'=i\,(\xi_8+\xi_9)$$

$$\xi_6'=\frac{i}{\sqrt{2}}\,(\xi_1+\xi_3-\xi_4-\xi_6) \qquad \xi_9'=-(\xi_6+\xi_{10})$$

$$\xi_7'=-i\,(\xi_7-\xi_{10}) \qquad \xi_{10}'=-i\,(\xi_8-\xi_9),$$

α^k becomes Kemmer's β matrix, so in these coordinate Bhabha's wave equation becomes Kemmer's type of 10 dimensions

$$\{\partial_\mu\,\beta_\mu+\varkappa\}\,\Phi=0,$$

where

$$p_1=i\,\frac{\partial}{\partial x^1},\quad p_2=i\,\frac{\partial}{\partial x^2},\quad p_3=i\,\frac{\partial}{\partial x^3},\quad p_4=i\,\frac{\partial}{\partial x^4}$$

$$\partial_1=\frac{\partial}{\partial x^1},\quad \partial_2=\frac{\partial}{\partial x^2},\quad \partial_3=\frac{\partial}{\partial x^3},\quad \partial_4=\frac{1}{i}\,\frac{\partial}{\partial x^4}$$

$$\beta^1=i\alpha',\quad \beta^1=i\alpha^2,\quad \beta^3=i\alpha^3,\quad \beta^4=-\alpha^4\,.$$

Thus Kemmer's wave functions are

$$\xi_1'=\frac{i}{\sqrt{2}}\,(\psi_{11}-\psi_{22}+\psi^{\dot{1}\dot{1}}-\psi^{\dot{2}\dot{2}}) \qquad \xi_6'=\frac{i}{\sqrt{2}}\,(\psi_{11}+\psi_{22}-\psi^{\dot{1}\dot{1}}-\psi^{\dot{2}\dot{2}})$$

$$\xi_2'=i\,(\psi_{12}+\psi^{\dot{1}\dot{2}}) \qquad \xi_7'=-i\,(\psi_1{}^{\dot{1}}-\psi_2{}^{\dot{2}})$$

$$\xi_3'=\frac{1}{\sqrt{2}}\,(\psi_{11}+\psi_{22}+\psi^{\dot{1}\dot{1}}+\psi^{\dot{2}\dot{2}}) \qquad \xi_8'=i\,(\psi_1{}^{\dot{2}}+\psi_2{}^{\dot{1}})$$

$$\xi_4'=\frac{1}{\sqrt{2}}\,(\psi_{11}-\psi_{22}-\psi^{\dot{1}\dot{1}}+\psi^{\dot{2}\dot{2}}) \qquad \xi_9'=-(\psi_1{}^{\dot{1}}+\psi_2{}^{\dot{2}})$$

$$\xi_{10}'=-i\,(\psi_1{}^{\dot{2}}-\psi_2{}^{\dot{1}})$$

$$\xi_5'=\psi_{12}-\psi^{\dot{1}\dot{2}}$$

Putting further

$$\boldsymbol{F}=(F_1,F_2,F_3)=(\xi_1',\xi_2',\xi_3')$$

$$\boldsymbol{G}=(G_1,G_2,G_3)=(\xi_4',\xi_5',\xi_6)$$

$$\boldsymbol{U}=(U_1,U_2,U_3)=(-\xi_7',-\xi_8',-\xi_9')$$

$$U^0=-\xi_{10}',$$

these equations become the ordinary equations of vector meson.

$$\frac{\partial\boldsymbol{F}}{\partial x^4}-\operatorname{curl}\boldsymbol{G}-\varkappa\boldsymbol{U}=0$$

$$\operatorname{div}\boldsymbol{F}+\varkappa U_0=0$$

$$\frac{\partial\boldsymbol{U}}{\partial x^4}+\operatorname{grad}U_0+\varkappa\boldsymbol{F}=0$$

$$\operatorname{curl}\boldsymbol{U}-\varkappa\boldsymbol{G}=0\,.$$

In the case of scalar meson field, we put for antisymmetrical spinors of the second rank

$$(\eta_1, \eta_2, \eta_3, \eta_4\,;\ \eta_5) = (\Psi_{13}, \Psi_{14}, \Psi_{23}, \Psi_{24}\,;\ \Psi_{12}+\Psi_{34})$$

The transformation $S_{ij}(\theta)$ of these quantity corresponding to the ij-plane Lorentz transformation by angle θ become as follows

$$S_{15}(\theta)=\left|\begin{array}{cccc|c} c^2 & 0 & 0 & s^2 & sc \\ 0 & c^2+s^2 & 0 & 0 & 0 \\ 0 & 0 & c^2+s^2 & 0 & 0 \\ s^2 & 0 & 0 & c^2 & -sc \\ \hline -sc & 0 & 0 & 0 & c^2-s^2 \end{array}\right|, \quad S_{25}(\theta)=\left|\begin{array}{cccc|c} c^2+s^2 & 0 & 0 & 0 & 0 \\ 0 & c^2 & -s^2 & 0 & -sc \\ 0 & -s^2 & c^2 & 0 & -sc \\ 0 & 0 & 0 & c^2+s^2 & 0 \\ \hline 0 & sc & sc & 0 & c^2-s^2 \end{array}\right|,$$

$$S_{35}(\theta)=\left|\begin{array}{cccc|c} c^2 & 0 & 0 & -s^2 & isc \\ 0 & c^2+s^2 & 0 & 0 & 0 \\ 0 & 0 & c^2+s^2 & 0 & 0 \\ -s^2 & 0 & 0 & c^2 & isc \\ \hline isc & 0 & 0 & isc & c^2-s^2 \end{array}\right|, \quad S_{45}(\theta)=\left|\begin{array}{cccc|c} c'^2-s'^2 & 0 & 0 & 0 & 0 \\ 0 & c'^2 & -s'^2 & 0 & -s'c' \\ 0 & -s'^2 & c'^2 & 0 & s'c' \\ 0 & 0 & 0 & c'^2-s'^2 & 0 \\ \hline 0 & -s'c' & s'c' & 0 & c'^2+s'^2 \end{array}\right|.$$

So α^k bebome

$$\alpha^1=\frac{1}{\sqrt{2}}\left|\begin{array}{cccc|c} & & & & 1 \\ & & & & 0 \\ & & 0 & & 0 \\ & & & & -1 \\ \hline -1 & 0 & 0 & 1 & 0 \end{array}\right|, \quad \alpha^2=\frac{1}{\sqrt{2}}\left|\begin{array}{cccc|c} & & & & 0 \\ & & & & -1 \\ & & 0 & & -1 \\ & & & & 0 \\ \hline 0 & 1 & 1 & 0 & 0 \end{array}\right|,$$

$$\alpha^3=\frac{1}{\sqrt{2}}\left|\begin{array}{cccc|c} & & & & i \\ & & & & 0 \\ & & 0 & & 0 \\ & & & & i \\ \hline i & 0 & 0 & i & 0 \end{array}\right|, \quad \alpha^4=\frac{1}{\sqrt{2}}\left|\begin{array}{cccc|c} & & & & 0 \\ & & & & -1 \\ & & 0 & & 1 \\ & & & & 0 \\ \hline 0 & -1 & 1 & 0 & 0 \end{array}\right|.$$

If we put

$$\eta_1'=\frac{i}{\sqrt{2}}(\eta_1-\eta_4)$$

$$\eta_2'=\frac{i}{\sqrt{2}}(\eta_2+\eta_3)$$

$$\eta_3'=\frac{-1}{\sqrt{2}}(\eta_1+\eta_4)$$

$$\eta_4'=\frac{i}{\sqrt{2}}(\eta_2-\eta_3)$$

$$\eta_5'=\eta_5,$$

α^k becomes Kemmer's β matrix, so in this coordinate Bhabha's wave equation becomes Kemmer's type of 5-dimensions.

Thus Kemmer's wave functions in this case are

$$\eta_1' = \frac{-i}{\sqrt{2}} (\psi_1^{\dot 1} - \psi_2^{\dot 2})$$

$$\eta_2' = \frac{i}{\sqrt{2}} (\psi_1^{\dot 2} + \psi_2^{\dot 1})$$

$$\eta_3' = \frac{1}{\sqrt{2}} (\psi_1^{\dot 1} + \psi_2^{\dot 2})$$

$$\eta_4' = \frac{i}{\sqrt{2}} (\psi_1^{\dot 2} - \psi_2^{\dot 1})$$

$$\eta_5' = \psi_{12} + \psi^{\dot 1 \dot 2} .$$

Putting further

$$\boldsymbol{U} = (U_1, U_2, U_3) = (\eta_1', \eta_2', \eta_3')$$
$$U_0 = \eta_4'$$
$$U = -\eta_5' ,$$

these equations become the ordinary equation of scalar meson:

$$\operatorname{div} \boldsymbol{U} + \frac{\partial U_0}{\partial x^4} - \varkappa U = 0$$

$$\operatorname{grad} U = \varkappa \boldsymbol{U}$$

$$\frac{\partial U}{\partial x^4} = -\varkappa U_0 .$$

In concluding, we wish to express our thanks to Professor K. Husimi for his kind interest and advices to our work.

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On the Lattice Polarization induced by Electronic Motion.

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§ 1. Introduction.

When an electron moves through the crystal lattice, the field of the electron induces the local electric polarization in the lattice, which polarization then affects in it's turn the motion of the electron. Such reaction of the lattice polarization on the electronic metion has usually been taken into account when necessary by the introduction of the empirical dielectric constant into the theory[1]. In what kind of circumstance is such phenomenological treatment will be justified? In the following, we should like to give a partial answer to this question. The comlete answer is no doubt only possible by a thorough quantum-mechanical treatment of the dielectricity of crystal lattices, which as far as we know, is not yet taken up by any investigators at the present time. In the present paper we have been able to enter into this important problem only superficially and we hope to come back again to the more detailed treatment of the subject.

In § 2 we give the method of the calculation of the polarization potential energy of an electron introduced into the ionic lattice, in § 3 we make some approximate calculations of this potential and discuss the trapping of an electron in a perfect ionic lattice, finally in § 4 we give some example of cases in which the usual theory with empirical dielectric constant does not work consistently.

§2. General Method of Calculation by Perturbation Theory

The Schrödinger equation of the system: an ionic lattice plus an additional electron is written as follows,

$$\left(-\frac{\hbar^2}{2m}\Delta+H_e+H_l+V+V'\right)\Psi=-\frac{\hbar}{i}\frac{\partial\Psi}{\partial t}, \qquad (2\cdot1)$$

where the first term on the left hand side is the kinetic energy of the additional electron, H_e and H_l are electronic and vibrational energy of the crystal lattice respectively. The interaction energy of the additional electron and the lattice V is composed from two parts, namely $V=V_1+V_2$, where V_1 is the electrostatic interaction with crystal electrons and ions

$$V_1=\sum_i e^2/|\vec{r}-\vec{r}_i|-S\sum_{l}\sum_{k} Z_k e^2/|\vec{r}-\vec{r}_k^{\,l}|\,, \tag{2·2}$$

in which $\vec{r}$ and $\vec{r}_i$ are the coordinates of the additional and the i-th lattice electron respectively and $-e$ is the electron charge, $\vec{r}_k^{\,l}$ is the equilibrium position of the k-th ion in the l-th unit cell, with its charge $Z_k e$, l is the abbreviation of the three cell-indices : $l=(l_1,\ l_2,\ l_3)$, V_2 is the interaction of the additional electron and the polarization field induced by the lattice vibration :

$$V_2=-e\Phi(\boldsymbol{r}), \tag{2·3}$$

where the polarization potential Φ is to be calculated by the Poisson equation from the electric polarization $\vec{P}(\boldsymbol{r})$:

$$\Delta\Phi=4\pi\, div\, \vec{P}(\vec{r}). \tag{2·4}$$

Finally V' is the similar interaction of the crystal electrons and the lattice polarization. Now the contributions to the polarization $\vec{P}(\boldsymbol{r})$ come mainly from the modes of lattice vibration of longer wave length, and as we concern in the following with the diatomic ionic lattices, we put

$$\vec{P}(\vec{r})=(Ze/\Delta)\,(\vec{u}_1^{\,l}-\vec{u}_2^{\,l}), \tag{2·5}$$

the unit cell of volume Δ contains two ions of charge Ze and $-Ze$, displacements of which are denoted here by $\vec{u}_1^{\,l}$ and $\vec{u}_2^{\,l}$ respectively. In (2·5) $\vec{u}_1^{\,l}-\vec{u}_2^{\,l}$ is assumed to be a continuous function of the position of the unit cell l.

For the harmonic vibrations of the lattice, displacement of each ion can be decomposed into the normal modes of vibration as follows

$$\vec{u}_k^{\,l}=1/\sqrt{G^3}\sum_{\vec{q}}\sum_{j}\vec{\varepsilon}^{\,(k)}_{\vec{q}j}\{a_{\vec{q}j}\exp i\vec{q}\vec{r}_k^{\,l}+\overset{*}{a}_{\vec{q}j}\exp -i\vec{q}\vec{r}_k^{\,l}\},\quad k=1,\ 2, \tag{2·6}$$

where G^3 is the total number of unit cells, $\vec{\varepsilon}^{\,(k)}_{\vec{q}j}$ is to be calculated from the unit vectors of polarization for the normal mode with the propagation vector $\vec{q}$, belonging to the j-th branch of vibration as follows,[2)]

for the optical branch, $j=1,\ 2,\ 3$,

$$\vec{\varepsilon}^{\,(1)}_{\vec{q}j}=(\sqrt{\mu}\,/M)\vec{e}_{\vec{q}j}\,,\qquad \vec{\varepsilon}^{\,(2)}_{\vec{q}j}=-(\sqrt{\mu}\,/M)\vec{e}_{\vec{q}j}\,, \tag{2·7}$$

and for the acoustical branch, $j=4,\ 5,\ 6$,

$$\vec{\varepsilon}^{\,(k)}_{\vec{q}j}=(1/M_1+M_2)^{1/2}\vec{e}_{\vec{q}j}\,,\quad k=1,\ 2, \tag{2·7'}$$

where M_1, M_2 are the masses of the positive and negative ion respectively and $\mu=(M_1M_2/M_1+M_2)$, further, $a_{\vec{q}j}$ and $a^*_{\vec{q}j}$ are the quantized amplitudes of the normal modes of vibration $(\vec{q}, j)$, of which only the following matrix elements differ from zero,

$$(N_{\vec{q}j} \,|\, a_{\vec{q}j} \,|\, N_{\vec{q}j}+1)=\sqrt{\hbar/2\omega_{\vec{q}j}(N_{\vec{q}j}+1)}, \tag{2·8}$$

$$(N_{\vec{q}j} \,|\, a^*_{\vec{q}j} \, N_{\vec{q}j}-1)=\sqrt{\hbar/2\omega_{\vec{q}j}N_{\vec{q}j}}, \tag{2·8'}$$

where $N_{\vec{q}j}$ is the number of phonons of mode $\vec{q}, j$. From (2·3), (2·4), (2·5), (2·6) we find

$$V_2=\frac{Ze^2}{\varDelta}\cdot\frac{4\pi i}{\sqrt{G^3\mu}}\sum_{\vec{q}}\frac{1}{q}\{a_{\vec{q}}\exp i\vec{q}\vec{r}-a^*_{\vec{q}}\exp -i\vec{q}\vec{r}\}, \tag{2·9}$$

and similarly

$$V'=\frac{Ze^2}{\varDelta}\frac{4\pi i}{\sqrt{G^3\mu}}\sum_{i}\sum_{\vec{q}}\frac{1}{q}\{a_{\vec{q}}\exp i\vec{q}\vec{r}_i-a^*_{\vec{q}}\exp -i\vec{q}\vec{r}_i\}. \tag{2·10}$$

Contribution to these potential energies comes only from the longitudinal waves of the optical mode, the amplitudes of which we now denote simply as $a_{\vec{q}}$ and $a^*_{\vec{q}}$.

As well known, the electric polarization of the ionic crystal lattices is induced from (1) the polarization of the charge cloud inside each ion and from (2) the relative displacements of the positive with respect to the negative ions. As will be seen from the following, in the language of the quantum-mechanical perturbation theory, these two mechanisms of the polarization can be explained to come from (1) the virtual excitation of the electronic states of the lattice and from (2) the virtual emission and reabsorption of phonons. It is noticeable, however, that in the higher stages of the perturbation calculations these two modes are mixed up and we can no longer separate them perfectly.

To solve the Schrödinger equation (2·1) approximately, we put

$$\varPsi=\psi(\vec{r})\,\varPhi(x_1, x_2, x_3, \cdots)\chi(Q)\exp -\frac{i}{\hbar}Wt, \tag{2·11}$$

where ψ, $\varPhi$ are the wave function of the additional electron and crystal electrons respectively and χ is the wave function of the lattice vibration. We neglect the exchange effect of the additional electron with the lattice electrons and x_i symbolizes the all coordinates of the i-th lattice electron, Q means the vibrational variables and W is the total energy. Inserting (2·11) in (2·1) and multiplying (2·1) from the left with $\varPhi^*\chi^*$, $\chi^*\psi^*$ and $\psi^*\varPhi^*$ respectively and integrating over the respective coordinates, we obtain three Hartree equations for the wave functions ψ, $\varPhi$ and χ, namely,

$$\{-\hbar^2/2m\Delta+\int\Phi^*\chi^*V\Phi\chi dxdQ-\varepsilon\}\psi(\vec{r})=0, \tag{2·12}$$

$$\{H_c+\int\chi^*\psi^*V\chi\psi dQd\vec{r}-\eta\}\Phi=0, \tag{2·13}$$

$$\{H_l+\psi^*\Phi^*V\psi\Phi d\vec{r}dx-\zeta\}\chi=0, \tag{2·14}$$

in which ε, η, ζ are the energy of the additional electron, lattice electrons and the lattice vibration respectively. For the ground state wave function of the lattice electrons Φ_0 and for the wave function of the N-th vibrational state χ_N we put

$$\Phi_0=\Phi_0^{(0)}+\Phi_0^{(1)}+\cdots, \tag{2·15}$$

and

$$\chi_N=\chi_N^{(0)}+\chi_N^{(1)}+\cdots,$$

then by the perturbation calculations we get from (2·13) and (2·14) the following expressions for Φ_0 and χ_N,

$$\Phi_0=\Phi_0^{(0)}+\int\psi^*\psi d\vec{r}\sum_n{}'\int\int\Phi_n^{(0)*}\chi_N^{(0)*}V\Phi_0^{(0)}\chi_N^{(0)}dxdQ/(\eta_0^{(0)}-\eta_n^{(0)})+\cdots, \tag{2·16}$$

$$\chi_N=\chi_N^{(0)}+\int\psi^*\psi d\vec{r}\sum_{N'}{}'\int\int\chi_{N'}^{(0)*}\Phi_0^{(0)*}V\chi_N^{(0)}\Phi_0^{(0)}dQdx/(\zeta_N^{(0)}-\zeta_{N'}^{(0)})+\cdots \tag{2·16′}$$

Thus substituting (2·16) and (2·16′) into Φ and χ of (2·12) we obtain the equation for $\psi(\vec{r})$:

$$\{-\hbar^2/2m\Delta+U_0+U_1+U_1^*-\varepsilon\}\psi(\vec{r})=0 \tag{2·17}$$

with

$$U_0(\vec{r})=\int\int\Phi_0^{(0)*}\chi_N^{(0)*}V\Phi_0^{(0)}\chi_N^{(0)}dx\,dQ,$$

$$U_1(\vec{r})=\int\int\Phi_0^{(0)*}\chi_N^{(1)*}V\Phi_0^{(0)}\chi_N^{(0)}dx\,dQ$$
$$+\int\int\Phi_0^{(1)*}\chi_N^{(0)*}V\Phi_0^{(0)}\chi_N^{(0)}dx\,dQ,$$

where the effective potential U_0 is that for the unpolarized lattice and $U_1+U_1^*$ denotes the reaction from the lattice polarization.

§ 3. Calculation of the Polarization Potential.

As is mentioned in § 2, substitution of (2·16) and (2·16′) into (2·19) separates the polarization potential $U_1(\boldsymbol{r})$ into two parts:

$$U_1+U_1^*=\int|\psi(\vec{r}')|^2\{U_v(\vec{r},\vec{r}')+U_e(\vec{r},\vec{r}')\}d\vec{r}', \tag{3·1}$$

in which $U_v(\vec{r},\vec{r}')$ which comes from virtual phonon emission is expressed as follows,

$$U_v(\vec{r}, \vec{r}') = \sum_{N'}{}' \left(\int |\Phi_0^{(0)}|^2 dx \int \chi_{N}^{(0)*} V_2(\vec{r}') \chi_{N'}^{(0)} dQ\right) \times$$

$$\times \left(\int |\Phi_0^{(0)}|^2 dx \int \chi_{N'}^{(0)*} V_2(\vec{r}) \chi_{N}^{(0)} dQ\right) / (\zeta_N^{(0)} - \zeta_{N'}^{(0)}) + conj.\ compl., \tag{3·2}$$

while $U_e(\vec{r}, \vec{r}')$ which comes from virtual electronic excitation is given by

$$U_e(\vec{r}, \vec{r}') = \sum_{n'}{}' \left(\int |\chi_N^{(0)}|^2 dQ \int \Phi_0^{(0)*} V_1(\vec{r}') \Phi_n^{(0)} dx\right) \times$$

$$\times \left(\int |\chi_N^{(0)}|^2 dQ \int \Phi_n^{(0)*} V_1(\vec{r}) \Phi_0^{(0)} dx\right) / (\eta_0^{(0)} - \eta_n^{(0)}) + conj.\ compl. \tag{3·3}$$

Substitution of (2·9) reduces (3·2) to the form

$$\left(\frac{Ze^2}{\Delta}\right)^2 \frac{(4\pi)^2}{2\mu G^3} \sum_{\vec{q}} \frac{1}{q^2\omega_q^2} \{N_{\vec{q}} \exp i\vec{q}, \vec{r} - \vec{r} - (N_{\vec{q}} + 1) \exp -i\vec{q}. \vec{r} - \vec{r}'\} + conj.\ compl.,$$

thus the terms with $N_{\vec{q}}$-factor cancel out and we find

$$U_v(\vec{r}, \vec{r}') = -\left(\frac{Ze^2}{\Delta}\right)^2 \frac{(4\pi)^2}{2\mu G^3} \sum_{\vec{q}} \frac{1}{q^2\omega_q^2} \{\exp i\vec{q}. \vec{r} - \vec{r}' + \exp -i\vec{q}. \vec{r} - \vec{r}'\}. \tag{3·4}$$

Making use of the Debye's approximation for the vibration spectrum, we put

$$\sum_{\vec{q}} \cdots = \frac{G^3 \Delta}{(2\pi)^3} \int\int_0^{\bar{q}} \cdots q^2 dq\, d\Omega\,, \quad \bar{q} = 2\pi(3/4\pi\Delta)^{1/3}, \quad d\Omega = solid\ angle\ element,$$

and further we neglect the change of optical mode-frequency with the wave length; putting $\omega_{\vec{q}} = \omega_0$ (Restsrahlfrequenz), we get

$$U_v(\vec{r}, \vec{r}') = -\frac{4\pi Z^2 e^2}{\Delta \mu \omega_0^2} \frac{e^2}{|\vec{r} - \vec{r}'|} \frac{2}{\pi} \int_0^{\bar{q}|\vec{r}-\vec{r}'|} \sin \xi / \xi\, d\xi. \tag{3·5}$$

On the other hand, (3·3) can be simply transformed to

$$U_e(\vec{r}, \vec{r}') = \sum_n{}' \left(\int \Phi_0^{(0)*} \sum_i e^2/|\vec{r}' - \vec{r}_i|\, \Phi_n^{(0)} dx\right) \left(\int \Phi_n^{(0)*} \sum_i e^2/|\vec{r} - \vec{r}_i|\, \Phi_0^{(0)} dx\right) / (\eta_0^{(0)} - \eta_n^{(0)})$$

$$+ compl.\ conj.. \tag{3·6}$$

Before getting into further reduction of (3·6) we notice that from (3·1), (3·5), (3·6) it is shown that the lattice polarization becomes the more prominent as the modulation of the electronic wave function $\psi(\vec{r})$ from the plane wave form becomes more and more stronger, until finally our perturbation calculation becomes meaningless for sufficiently large polarization. We should like to remark further that $U_v(\vec{r}, \vec{r}')$ behaves for $|\vec{r} - \vec{r}'| \gg$ *lattice constant* as $|\vec{r} - \vec{r}'|^{-1}$ and becomes constant for $|\vec{r} - \vec{r}'| \ll$ *lattice constant* (Fig. 1), but we must remind that in our treatment, the detailed behaviour of the potential inside the unit cell has been neglected from the beginning (cf. (2·5)) and thus the constant value must be interpreted as the average value of $U_v(\vec{r},\vec{r}')$ in the unit cell.

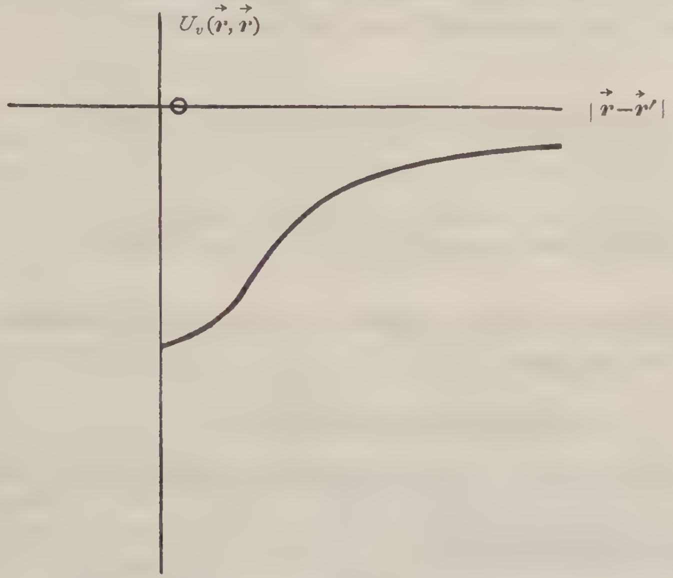

Fig. 1.

To make further reduction of (3·6) we construct the wave functions of the crystal electrons by the Bloch's tightly bound orbitals:

$$\phi_{\vec{k}s}=1/\sqrt{G^3}\,\underset{l}{S}\,\phi_s(\vec{r}-\vec{r}_k^l)\exp i\vec{k}\vec{r}_k^l, \tag{3·7}$$

where ϕ_s is an ionic orbital, $\vec{k}$ is the propagation vector and S_l means summation with respect to the cell-index l. Neglecting the overlappings of the different ionic orbitals, and assuming that the most prominent contributions come from the ionic dipole transitions, we get after some simple reductions

$$U_e(\vec{r},\vec{r}')=\frac{(4\pi)^2}{G^6\Delta^2}\sum_{\vec{k}}\sum_{\vec{k}'}\exp i(\vec{k}-\vec{k}',\vec{r}-\vec{r}')/|\vec{k}-\vec{k}|^2\times$$

$$\times 2e^4\sum_{s}\sum_{s'}|\int\phi^*_{s'}(\vec{\rho})\vec{\rho}\phi_s(\vec{\rho})d\vec{\rho}|^2/(\eta^{(0)}_{\vec{k}s}-\eta^{(0)}_{\vec{k}'s'}), \tag{3·8}$$

where the summations with respect to s and s' are to be carried out on the filled and the empty bands respectively. Disregarding the energy spread of each band we put $\eta^{(0)}_{\vec{k}s}-\eta^{(0)}_{\vec{k}'s'}\approx\eta^{(0)}_s-\eta^{(0)}_{s'}$; we can further simplify (3·8) to the following form,

$$U_e(\vec{r}\ \vec{r}')=2\frac{4\pi e^2}{\Delta}\Big(\sum_{s}\sum_{s'}|\int\phi^*_{s'}\rho_z\phi_s|^2/(\eta^{(0)}_s-\eta^{(0)}_{s'})\Big)e^2/|\vec{r}-\vec{r}'|\cdot 2/\pi\int_0^{\bar{q}|\vec{r}-\vec{r}'|}\sin\xi/\xi\,d\xi\,, \tag{3·9}$$

with $\bar{q}=2\pi(3/4\pi\Delta)^{1/3}$.

Remembering that $p_l=Z^2e^2/\Delta\mu\omega_0^{2(3)}$, and $p_e=2e^2/\Delta\sum_{s}\sum_{s'}|\int\phi^*_{s'}\rho_z\phi_s|^2/(\gamma_s^2-\gamma_{s'}^2)$ are the polarizability per unit volume of the lattice, each arising from the ionic displacement and the polarization inside each ion respectively, (2·17) can be written in the form

$$\{-\hbar^2/2m\Delta+U_0(\vec{r})-\int|\psi(\vec{r}')|^2F(\vec{r}-\vec{r}')d\vec{r}'-\varepsilon\}\psi(\vec{r})=0\,, \qquad (2\cdot17')$$

with

$$F(\vec{r}-\vec{r}')=4\pi(p_l+p_e)2/\pi\cdot e^2/|\vec{r}-\vec{r}'|\int_0^{\bar{q}|\vec{r}-\vec{r}'|}\sin\xi/\xi\,d\xi\,, \qquad (3\cdot10)$$

and $U_0(\vec{r})$ is the periodic lattice potential. In the limit of continous medium we put $\Delta\rightarrow0$, $\bar{q}\rightarrow\infty$ and get

$$F(\vec{r}-\vec{r}')\rightarrow(\varkappa-1)e^2/|\vec{r}-\vec{r}'|\,, \qquad (3\cdot11)$$

if we put $\varkappa=1+4\pi(p_l+p_e)$. $\varkappa$ is the dielectric constant of the lattice for low frequencies.

In this limit (2·17) becomes

$$\{-\hbar^2/2m\Delta+U_0-(\varkappa-1)\int e^2/|\vec{r}-\vec{r}'|\cdot|\psi(\vec{r}')|^2d\vec{r}'-\varepsilon\}\psi(\vec{r})=0. \qquad (3\cdot13)$$

Similar equation:

$$\{-\hbar^2/2m\Delta+U_0-(1/\varkappa_0-1/\varkappa)\int e^2/|\vec{r}-\vec{r}'|\cdot|\psi(\vec{r}')|^2d\vec{r}'-\varepsilon\}\psi(\vec{r})=0, \qquad (3\cdot14)$$

$$\varkappa_0=1+4\pi p_e,$$

was once proposed by Gurney and Mott[4] in their discussions about the possibility of the self-trapping of an electron in a perfect ionic lattice, which possibility had been suggested by L. Landau.[5] Our result (3·13) differs from (3·14) as to the coefficient of the integral, but our perturbation method presupposes, strictly speaking, that $\varkappa-1,\ \varkappa_0-1\ll1$ and in this limiting case the discrepancy becomes unnoticeable.

As to the validity of our perturbation method, it should be remarked, that the contribution of the polarization potential to the wave function $\psi(\vec{r})$ and the energy ε must be smaller than that from $U_0(\vec{r})$. For the above mentioned problem of the electron trapping, such condition will be realized if the binding energy of the trapped electron is much smaller than the distance between the energy bands. Further, the detailed structure of the polarization potential in the unit cell is not taken into account in our theory and this fact introduce neglisible errors only for the widely extended wave functions, namely for the excited states of the trapped electron.

§ 4. Aperiodic Processes in the Crystal Lattice.

In the preceding paragraphs we have discussed the reaction of lattice polarization for the stationary states of an electron. In this paragraph we shall further concern ourselves with the influence of the lattice polarization to the transition probability. We take two examples;

(A) Excitation of the optical modes of vibration.

The probability of the process in which an electron with momentum $\vec{p}$ collides with the lattice and excites a quantum of longitudinal vibration of the optical mode of wave number $\vec{q}$ and deflected with the momentum $\vec{p}'$ is determined by the following matrix element,

$$\begin{aligned}&(\vec{p}', N_{\vec{q}}+1 \mid V_2 \mid \vec{p}, N_{\vec{q}})\\ &\quad+\sum_n{}'(N_{\vec{q}}+1, 0 \mid V' \mid (N_{\vec{q}}, n)(\vec{p}', n \mid V_1 \mid \vec{p}, o/[E(p)-E(p')+\eta_0^{(0)}-\eta_n^{(0)}]\\ &\quad+\sum_n{}'(\vec{p}', o \mid V_1 \mid \vec{p}, n)(N_{\vec{q}}+1, n \mid V' \mid N_{\vec{q}}, o)/[\eta_0^{(0)}-\eta_n^{(0)}-\hbar\omega_q], \qquad (4\cdot1)\end{aligned}$$

where $E(p)=p^2/2m$, the first term comes from the direct transition and the second and the third terms come through the virtual excitation of the electronic states of the crystal. Taking account of the energy conservation $E(p)-E(p')=\hbar\omega_q$ and the inequality $\hbar\omega_q \ll \eta_n^{(0)}-\eta_0^{(0)}$, which condition is generally satisfied for ionic crystals, we can reduce (4·1) to the simpler expression

$$(\vec{p}', N_{\vec{q}}+1 \mid V_2 \mid \vec{p}, N_{\vec{q}})\left\{1+\frac{4\pi}{\Delta}\sum_s\sum_{s'} 2e^2 \left|\int \phi_{s'}^* \rho_z \phi_s\right|^2/(\eta_s^{(0)}-\eta_{s'}^{(0)})\right\}, \qquad (4\cdot2)$$

where the same approximation is used for the electronic eigenfunctions as in § 3. (4·2) shows that the effect of the polarization can be taken into account by the factor $(1-4\pi p_e)$ which in the case $p_e \ll 1$ can be expressed by the dielectric constant $\varkappa_0$, namely $(2-\varkappa_0)$.

(B) Excitation of the lattice excitation waves.

Wave functions of the lattice excitation waves are written as

$$\psi_{\vec{K},n}^{(k)}=1/\sqrt{G^3}\,\underset{l}{S}\,\psi_n^{(k)} \exp i\vec{K}\vec{r}_k^l, \qquad (4\cdot3)$$

where $\psi_n^{(k)}$ describes the state in which the k-th ion in the l-th unit cell is excited in its n-th state, and $\vec{K}$ is the wave vector of the excitation wave. The matrix element for the process, in which the momentum of the additional electron changes from $\vec{p}$ to $\vec{p}'$ while the $(\vec{K}, n)$ excitation wave is excited, comes out to be

$$\begin{aligned}&(\vec{p}', \vec{K}n \mid V_1 \mid \vec{p}, 0)+\sum_{\vec{q}}(N_{\vec{q}}, \vec{K}n \mid V' \mid N_{\vec{q}}+1, 0)(\vec{p}', N_{\vec{q}}+1 \mid V_2 \mid \vec{p}, N_{\vec{q}})/[E(p)\\ &-E(p')-\hbar\omega_q]+\sum_{\vec{q}}(N_{\vec{q}}, \vec{K}n \mid V' \mid N_{\vec{q}}-1, 0)(\vec{p}', N_{\vec{q}}-1 \mid V_2 \mid \vec{p}, N_{\vec{q}})/[E(p)\end{aligned}$$

$$-E(p')+\hbar\omega_q]+\sum_{\vec{q}}(\vec{p}';N_{\vec{q}}|V_1|\vec{p},N_{\vec{q}}+1)(N_{\vec{q}}+1,\vec{K}n|V'|N_{\vec{q}},0)/[\eta_0$$

$$-\eta_{\vec{K},n}-\hbar\omega_q]+\sum_{\vec{q}}(\vec{p}',N_{\vec{q}}|V_2|\vec{p},N_{\vec{q}}-1)(N_{\vec{q}}-1,\vec{K}n|V'|N_{\vec{q}},0)\div$$

$$[\eta_0-\eta_{\vec{K},n}+\hbar\omega_q], \tag{4·4}$$

in which $\eta_{\vec{K}u}$ is the energy of the excitation wave and the first term comes from the direct transition while the other terms come through the virtual excitation of the lattice vibrations. Further reduction of the formula becomes possible by the fact that excitation energies of the excitation waves are generally greater than $\hbar\omega_q$. Thus (4·4) is reduced approximately to

$$(\vec{p}'\vec{K}n|V_1|\vec{p},0)\{1+(4\pi Z^2e^2/\Delta\mu\omega_0^2)(\hbar\omega_0/\eta_{\vec{K}n}-\eta_0)\}. \tag{4·5}$$

Unlike the case of example (A) the cerrection factor in (4·5) cannot be simply expressed by the dielectric constant, moreover, here, contrary to the simple phenpmenological expectations, the influence of the lattice polarization does not reduce but increase the transition probability. Such difference in the above two examples (A) aud (B) is, as can be easily recognized, caused by the difference of the relative magnitudes of the energy change in the transition processes and the virtual excitation energy.

Conclusion.

We have developed a quantum-mechanical perturbation calculation for the reaction of the lattice polarization on the electronic motion and it is found out that the mechanism of the electron trapping in a perfect ionic lattice as proposed by Landau, Gurney and Mott is also expected from our quantum-mechanical view point, though our result has not been in perfect agreement with that of above authors. Further the influence of the lattice polarization on the probability of electronic transitions which take place in an ionic lattice has been examined by two examples and it is pointed out that such influence can not be always accounted for by the simple phenomenological introduction of the empirical dielectric constant.

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On an Approximate Solution of the Many Body Problem of Fermi-Dirac Particles.

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1. Introduction

As well known, the strong coupling between the motions of individual nucleons makes it extremely difficult to solve the nuclear many body problem on the rational quantum-mechanical basis. Liquid drop model, which has furnished with many remarkably good descriptions of nuclear properties remains thus far entirely phenomenological and must be drastically reformed to give satisfactory interpretations of nuclear phenomena in terms of the nuclear forces.

The familiar Hartree-Fock's method, on the other hand, especially for the heavy nuclei, does not converge rapidly and so must be worked out, even for the purpose of qualitative discussions, to its higher oder of approximations. These situations are clearly recognized in the calculations of Euler,[1] Bagge[2] and others, in which the second order of approximation was explicitely worked out for the nuclear binding energy and the wave function of the ground state.

In the present paper we give a solution of the Schrödinger equation for the interacting Fermi-Dirac particles, which for the case of sufficiently large number of particles may be a good approximation and will be useful as a plausible starting point for the further stages of the perturbation calculations. In § 2 we give the general theory and its application to the nuclear problems is worked out in § 3.

2. General Method of Calculation

We start from the Hamiltonian[3]

$$\bar{H}=\int\tilde{\boldsymbol{\phi}}(x)\,H(x)\,\boldsymbol{\phi}(x)\,dx+\frac{1}{2}\iint\tilde{\boldsymbol{\phi}}(x)\,\tilde{\boldsymbol{\phi}}(x')\,V(x,x')\,\boldsymbol{\phi}(x')\,\boldsymbol{\phi}(x)\,dx\,dx' \qquad (2\cdot1)$$

for the system of interacting Fermi-Dirac particles, where $H(x)$ and $V(x,x')$ are respectively the one particle energy operator and the operator of interaction potential and $\boldsymbol{\phi}(x), \tilde{\boldsymbol{\phi}}(x)$ are the q-number amplitudes satisfying the following commutation relations,

$$\tilde{\boldsymbol{\phi}}(x)\,\boldsymbol{\phi}(x')+\boldsymbol{\phi}(x')\,\tilde{\boldsymbol{\phi}}(x)=\delta(x-x'), \qquad (2\cdot2)$$

$$\boldsymbol{\phi}(x)\boldsymbol{\phi}(x')+\boldsymbol{\phi}(x')\boldsymbol{\phi}(x)=\tilde{\boldsymbol{\phi}}(x)\tilde{\boldsymbol{\phi}}(x')+\tilde{\boldsymbol{\phi}}(x')\tilde{\boldsymbol{\phi}}(x)=0, \tag{2·3}$$

or if we expand $\boldsymbol{\phi}(x)$ by the set of c-number eigenfunctions $\psi_i(x)$ of $H(x)$ as

$$\boldsymbol{\phi}(x)=\sum_i \boldsymbol{a}_i\psi_i(x), \quad \{H(x)-E_i\}\psi_i(x)=0, \tag{2·4}$$

we find

$$\tilde{\boldsymbol{a}}_i\boldsymbol{a}_j+\boldsymbol{a}_j\tilde{\boldsymbol{a}}_i=\delta_{ij}, \ \boldsymbol{a}_i\boldsymbol{a}_j+\boldsymbol{a}_j\boldsymbol{a}_i=\tilde{\boldsymbol{a}}_i\tilde{\boldsymbol{a}}_j+\tilde{\boldsymbol{a}}_j\tilde{\boldsymbol{a}}_i=0: \tag{2·5}$$

thus $\tilde{\boldsymbol{a}}_i\boldsymbol{a}_i=N_i$ is the number of particles in the state i and $\boldsymbol{a}_i$ and $\tilde{\boldsymbol{a}}_i$ are the annihilation and the creation operators respectively.

Substituting (2·4) in $\boldsymbol{\phi}(x)$ of (2·1), (2·1) is transformed to

$$\bar{H}=\sum_i E_iN_i+\frac{1}{2}\sum_i\sum_k(V_{ik,ki}-V_{ik,ik})N_iN_k+\frac{1}{2}\sum_l{}'\sum_m{}'\sum_i\sum_k V_{ik,lm}\tilde{\boldsymbol{a}}_i\tilde{\boldsymbol{a}}_k\boldsymbol{a}_l\boldsymbol{a}_m, \tag{2·6}$$

where the matrix element $V_{ik,lm}$ is defined by

$$V_{ik,lm}=\iint\tilde{\psi}_i(x)\tilde{\psi}_k(x')\,V(x,x')\,\psi_l(x')\,\psi_m(x)\,dx\,dx', \tag{2·7}$$

and dashes at the summation signs mean the omission of the terms ($l=i$, $m=k$) and ($l=k$, $m=i$). If $\psi_i(x)$ are the plane waves, for which case we should like to confine ourselves in the following, the terms like ($l=i$, $k\neq m$) and ($l\neq i$, $k=m$) are, however, automatically excluded by the conservation of momentum and the multiple summations in the last term of (2·6) are to be extended only to the terms such that both l and m are not equal to i nor k.

If we start from the ground state of the system, where all levels $r, s, t, \ldots\ldots$ below the Fermi surface are completely filled with particles, while all levels $i, k, l, \ldots\ldots$ above the Fermi surface are all vacant, the last term of (2·6) ("*collision term*") induces collision between any two particles below the Fermi surface and sends these two particles in the upper vacant levels. However, as thus the population of particles above and of the holes below the Fermi surface is increased, this "*collision term*" induces further the collisions, in which at least one of the created holes or the ejected particles themselves takes part. To investigate these various modes of collisions separately, we rearrange the Hamiltonian (2·6) in the following manner,

$$\bar{H}=\bar{H}_0+\bar{H}'+\bar{H}'', \tag{2·8}$$

$$\begin{aligned}\bar{H}_0=&\mathop{S}_r E_rN_r+\sum_t E_tN_t+\frac{1}{2}\mathop{S}_r\mathop{S}_s(V_{rs,sr}-V_{rs,rs})N_rN_s+\frac{1}{2}\mathop{S}_r\sum_t(V_{rt,tr}-V_{rt,rt})N_rN_t\\&+\frac{1}{2}\sum_i\mathop{S}_r(V_{ir,ri}-V_{ir,ir})N_iN_r+\frac{1}{2}\sum_i\sum_k(V_{ik,ki}-V_{ik,ik})N_iN_k,\end{aligned} \tag{2·9}$$

$$H=\frac{1}{2}\sum_i\sum_k\mathop{S}_r\mathop{S}_s V_{ik,rs}\tilde{\boldsymbol{a}}_i\tilde{\boldsymbol{a}}_k\boldsymbol{a}_r\boldsymbol{a}_s+\frac{1}{2}\sum_i\sum_k\mathop{S}_r\mathop{S}_s V_{rs,ik}\tilde{\boldsymbol{a}}_r\tilde{\boldsymbol{a}}_s\boldsymbol{a}_i\boldsymbol{a}_k, \tag{2·10}$$

$$\begin{aligned}\bar{H}'' = &\frac{1}{2}\sum_i\sum_k\sum_l\sum_m V_{ik,lm}\tilde{\boldsymbol{a}}_i\tilde{\boldsymbol{a}}_k\boldsymbol{a}_l\boldsymbol{a}_m + \frac{1}{2}\mathop{S}_r\mathop{S}_s\mathop{S}_t\mathop{S}_u V_{rs,tu}\tilde{\boldsymbol{a}}_r\tilde{\boldsymbol{a}}_s\boldsymbol{a}_t\boldsymbol{a}_u \\ &+\mathop{S}_r\mathop{S}_s\sum_i\sum_k (V_{ri,sk}-V_{ri,ks})\tilde{\boldsymbol{a}}_r\tilde{\boldsymbol{a}}_i\boldsymbol{a}_s\boldsymbol{a}_k \\ &+\frac{1}{2}\mathop{S}_r\sum_i\sum_k\sum_l (V_{ri,kl}-V_{ir,kl})\tilde{\boldsymbol{a}}_r\tilde{\boldsymbol{a}}_i\boldsymbol{a}_k\boldsymbol{a}_l \\ &+\frac{1}{2}\mathop{S}_r\sum_i\sum_k\sum_l (V_{ik,rl}-V_{ik,lr})\tilde{\boldsymbol{a}}_i\tilde{\boldsymbol{a}}_k\boldsymbol{a}_r\boldsymbol{a}_l \\ &+\frac{1}{2}\sum_i\mathop{S}_r\mathop{S}_s\mathop{S}_t (V_{ir,st}-V_{ri,st})\tilde{\boldsymbol{a}}_i\tilde{\boldsymbol{a}}_r\boldsymbol{a}_s\boldsymbol{a}_t \\ &+\frac{1}{2}\sum_i\mathop{S}_r\mathop{S}_s\mathop{S}_t (V_{rs,it}-V_{rs,ti})\tilde{\boldsymbol{a}}_r\tilde{\boldsymbol{a}}_s\boldsymbol{a}_i\boldsymbol{a}_t, \qquad (2\cdot11)\end{aligned}$$

where the summations $\mathop{S}_r$ and $\sum_i$ are to be extended to the levels below and above the Fermi surface respectively.

(2·9) is the diagonal energy, (2·10) contains the collision terms which induce the collisions between two particles below the Fermi surface and their reverse collisions. Termes in (2·11), on the other hand, all concern with the collisions in which at least one hole or one ejected particle takes part. Now, if the total number of particles in the system is infinitely large, number of possible combinations of the partners of collisions in which at least one of the partners is a hole or an ejected particle remains, in any given stage of higher order perturbation calculation, infinitely less than the number of combinations in which both partners of the collision are the particles which still remain below the Fermi surface. Because of this situation it seems possible, for the system composed of sufficiently many particles, first to disregard entirely the operator $\bar{H}''$ and solve the problem $\bar{H}_0+\bar{H}'$. Certainly we must notice, however, that if the matrix elements like $V_{rs,ik}$ favour strongly the transitions with minor energy changes then there appears, even for the case of infinitely many particles (and thus for the infinite depth of the level distribution below the Fermi surface) the effective depth of energy below the Fermi surface such that only above which the transitions will practically take place. In this case our omission of $\bar{H}''$ becomes less plausible but we find no way to solve the difficulty except by taking account of $\bar{H}''$ further by the perturbation method.

Now, in the following, we should like to show that the same assumption of the diluteness of the holes and the ejected particles leads to an approximate solution of the problem $\bar{H}_0+H'$.

We first make the transformation

$$\boldsymbol{a}_r=\tilde{\boldsymbol{b}}_r, \qquad \tilde{\boldsymbol{a}}_r=\boldsymbol{b}_r,$$

and thus introduce the number of holes $\tilde{\boldsymbol{b}}_r\boldsymbol{b}_r=n_r$ below the Fermi surface.

Then we can rewrite $\overline{H}_0$ and $\overline{H}'$ as follows,

$$\overline{H}_0=\mathop{S}_{r}E_r+\frac{1}{2}\mathop{S}_{r}\mathop{S}_{s}(V_{sr,rs}-V_{sr,sr})+\sum_{i}E_iN_i-\mathop{S}_{r}E_rn_r$$
$$+\sum_{i}\mathop{S}_{r}(V_{ir,ri}-V_{ir,ir})N_i-\mathop{S}_{r}\mathop{S}_{s}(V_{sr,rs}-V_{sr,sr})n_r$$
$$+\frac{1}{2}\sum_{i}\sum_{k}(V_{ik,ki}-V_{ik,ik})N_iN_k-\sum_{i}\mathop{S}_{r}(V_{ir,ri}-V_{ir,ir})N_in_r$$
$$+\frac{1}{2}\mathop{S}_{r}\mathop{S}_{s}(V_{sr,rs}-V_{sr,sr})n_rn_s, \tag{2·13}$$

$$\overline{H}'=\frac{1}{2}\sum_{i}\sum_{k}\mathop{S}_{r}\mathop{S}_{s}V_{ik,rs}\tilde{a}_i\tilde{a}_k\tilde{b}_r\tilde{b}_s+\frac{1}{2}\sum_{i}\sum_{k}\mathop{S}_{r}\mathop{S}_{s}V_{rs,ik}b_rb_sa_ia_k \tag{2·14}$$

The first two terms of (2·13) are the energies of the system for the complete degeneracy and the following terms denote the deviations induced by the creation of holes, among which we shall in the following neglect the terms which are quadratic in N or n in accordance with the above mentioned "*diluteness assumption*".

Thus we are now to solve the following Schrödinger equation

$$(W-\overline{H}_0^{(0)}-\sum_{i}\varepsilon_iN_i+\mathop{S}_{r}\varepsilon_rn_r)\Psi(\ldots N_i\ldots;\ \ldots n_r\ldots)$$
$$=\overline{H}'\Psi(\ldots N_i\ldots;\ \ldots n_r\ldots), \tag{2·15}$$

with $\overline{H}_0^{(0)}=\mathop{S}_{r}E_r+\frac{1}{2}\mathop{S}_{r}\mathop{S}_{s}(V_{sr,rs}-V_{sr,sr})$, $\varepsilon_i=E_i+\mathop{S}_{r}(V_{ir,ri}-V_{ir,ir})$,

$$\varepsilon_r=E_r+\mathop{S}_{r}(V_{rs,sr}-V_{rs,rs}).$$

To find out the ground state of the system we write down (2·15) in the following form

$$(W-\overline{H}_0^{(0)})\Psi(0)=\sum_{i>k}\sum\mathop{S}_{r>s}\mathop{S}U_{ik,rs}\Psi(ik;\ rs), \tag{2·16}$$

$$(W-\overline{H}_0^{(0)}-\varepsilon_{ik,rs})\Psi(ik;\ rs)=U_{rs,ik}\Psi(0)$$
$$+\sum_{i'>k'}\sum\mathop{S}_{r'>s'}\mathop{S}U_{i'k',r's'}\Psi(ik,rs;\ i'k',r's'), \tag{2·17}$$

$$(W-\overline{H}_0^{(0)}-\varepsilon_{ik,rs}-\varepsilon_{i'k',r's'})\Psi(ik,rs;\ i'k',r's')=U_{r's',i'k'}\Psi(ik,rs)$$
$$+U_{rs,ik}\Psi(i'k',r's')+\cdots+\sum_{i''>k''}\sum\mathop{S}_{r''>s''}\mathop{S}U_{i''k'',r''s''}\Psi(ik,rs;i'k',r's';i''k'',r''s''),$$

....

$$\tag{2·18}$$

where $\varepsilon_{ik,rs}=\varepsilon_i+\varepsilon_k-\varepsilon_r-\varepsilon_s$, $U_{rs,ik}=\eta_r\eta_s\eta_i\eta_k(V_{rs,ik}-V_{sr,ik})$, with $\eta_i=(-1)^{N_1+N_2\ldots+N_i}$ and $\Psi(\ldots1_i\ldots1_k\ldots;\ \ldots1_r\ldots1_s\ldots)$ is abbreviated as $\Psi(ik,rs)$. The dotts in the right hand side of (2·18) is written in place of the terms such as $U_{rs',ik'}\Psi(i'k, r's)$, $U_{r's,i'k}\Psi(ik', rs')$ etc. In the state $\Psi(ik, rs;\ i'k', r's')$ we have four particles i, k, l', k' above and four holes r, s, r', s' below the Fermi surface, of which i, k, r, s and i', k', r', s' are such groupings that momentums are conserved among the members of each group. However it is possible that momentum conservation is simultaneously satisfied also by such groupings as i', k, r, s; i, k', r, s'; ... and in that case we must take into account of the terms represented by the dotts in the right hand side of (2·18). All the equations following (2·18) contain similar terms on their right hand sides, which fact makes it very difficult to solve these simultaneous equations exactly. In the case of infinitely large particle number, however, these terms can be neglected without causing serious errors and then our simultaneous equations take the form which can be solved rigorously.* The solution is found to be as follows.

$$\begin{aligned}
&\Psi(ik, rs) = -\Psi(0)\,U_{rs,ik}/\varepsilon_{ik,rs},\\
&\Psi(ik, rs;\ i'k', r's') = \Psi(0)\,(U_{rs,ik}/\varepsilon_{ik,rs})\,(U_{r's',i'k'}/\varepsilon_{i'k',r's'}), \qquad (2\cdot19)\\
&\Psi(ik, rs;\ i'k', r's';\ i''k'', r''s'') = -\Psi(0)\,(U_{rs,ik}/\varepsilon_{ik,rs})\,(U_{r's',i'k'}/\varepsilon_{i'k',r's'})\times\\
&\cdots\cdots\cdots \qquad \times(U_{r''s'',i''k''}\ \varepsilon_{i''k'',r''s''}),
\end{aligned}$$

and

$$W = \overline{H}_0^{(0)} - \sum_{i>k}\sum\ \underset{r>s}{S\,S}\,|U_{ik,rs}|^2/\varepsilon_{ik,rs}. \qquad (2\cdot20)$$

It is to be remarked that the essential assumption in our treatment is not the smallness of the coupling between the particles as in the usual theories, but the largeness of the total number of particles contained in the system. The energy of the ground state (2·20)

$$W = \overline{H}_0^{(0)} - \sum_{i>k}\sum\ \underset{r>s}{S\,S}\,|U_{ik,rs}|^2/[\varepsilon_i+\varepsilon_k-\varepsilon_r-\varepsilon_s], \qquad (2\cdot21)$$

however, comes out to be very similar to the usual second order result of the perturbation theory and differs from it only by the energy denominator in which, in our case, are contained the coupling potentials as can be seen from the definition of ε_i. Further, the distribution of particles on the energy levels in the grouud state is found to be markedly different from that given by the zeroth approximation, namely all levels are occupied below the Fermi surface and all levels are vacant above this surface, but rather resembles, as has been especially remarked by S. Watanabe,[4] to the particle distribution for the finite temperature, which situation may be quantitatively computable by the evaluation $|\Psi(ik, rs)|^2$, $|\Psi(ik, rs;\ i'k', r's')|^2$, $|\Psi(ik, rs;\ i'k', r's';\ i''k'', r''s'')|^2$ from (2·19).

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* These equations are of the same type as those, which are solved by S. Tomonaga in his theory of the neutral longitudinal mesons surrounding a nucleon, cf. his following paper, Prog. Theor. Phys. **2** (1947), 6.

§ 3. Application to the Heavy Nuclei

Among the possible applications of the general theory developed in the preceding paragraphs we shall confine ourselves in the following to the binding energy of the heavy atomic nuclei.

Nuclear interaction potential is given by

$$V_{12}=V(r_{12})(C_W-C_M\boldsymbol{PQ}-C_H\boldsymbol{P}+C_B\boldsymbol{Q}), \tag{3·1}$$

with $V(r_{12})=-V\exp-(r_{12}/a)^2$; $\boldsymbol{P}, \boldsymbol{Q}$ are the exchange operators for the isotopic spin and spin respectively and c_W, c_M, c_H, c_B are usual coefficients for Wigner-Majorana-, Heisenberg-, and Bartlett forces. ε in (2·15) is now characterized with the quantum numbers of momentum $\vec{p}$, of spin and isotopic spin σ, τ and expressed as

$$\varepsilon(\vec{p},\sigma,\tau)=E(p)+C'\left[\int V(r_{12})\,d\vec{r}_{12}\right]1/V_0\,\underset{\vec{p}'}{S^{oc}}$$

$$-C/V_0\,\underset{\vec{p}'}{S^{oc}}\int V(r_{12})\exp\frac{i}{\hbar}(\vec{p}-\vec{p}',\vec{r}_{12})\,d\vec{r}_{12}, \tag{3·2}$$

in which $E(p)$ is the energy of the free wave $\vec{p}$, V_0 is the nulear volume and $\underset{\vec{p}'}{S^{oc}}$ denotes the summation with respect to $\vec{p}'$ on all occupied levels. Further, $c'=4c_W-c_M-2c_H+2c_B$ and $c=c_W-4c_M-2c_H+2c_B$.

In (2·21) enters only the difference of $\varepsilon(\vec{p},\sigma,\tau)$ and thus the second term of (3·2) makes no contribution while the third term can be transformed by the aid of the relation

$$\underset{\vec{p}'}{S^{oc}}\exp-\frac{i}{\hbar}\vec{p}'\vec{r}=V_0/2\pi^2\cdot1/\lambda_0^3(\lambda_0/r)^2(\pi r/2\lambda_0)^{1/2}J_{3/2}(r/\lambda),$$

thus (3·2) becomes

$$\varepsilon(\vec{p},\sigma,\tau)=E(p)+c\lambda/\lambda_0\sqrt{\lambda\lambda_0}\cdot V\int_0^\infty\exp-(r/a)^2\cdot J_{3/2}(r/\lambda_0)\,J_{3/2}(r/\lambda)\,dr, \tag{3·3}$$

where $\lambda=\hbar/p$ and λ_0 is the minimum wave length of the particle inside the nucleus, $J_{1/2}$, $J_{3/2}$ are the Bessel functions. The integral in (3·3) should have been extended inside the nucleus but the range of integration can be without serious errors enlarged to the positive infinity. This definite integral can be evaluated without difficulty, but for the limiting case $a/\lambda_0>1$ (according to Euler and Bagge $2·2<a/\lambda_0<2·75$) we can simply replace this integral by

$$\gamma\int J_{3/2}(r/\lambda_0)\,J_{3/2}(r/\lambda)\,dr=\begin{cases}\lambda_0\sqrt{\lambda\lambda_0}/\lambda, & \lambda>\lambda_0\\ 0 & \lambda>\lambda_0\end{cases},\quad \gamma\gtrsim 1, \tag{3·4}$$

thus we find

$$\varepsilon(p,\sigma,\tau)=E(p)+\gamma C\bar{V}+constant \quad \lambda>\lambda_0,$$
$$\varepsilon(p,\sigma,\tau)=E(p) \qquad\quad +constant \quad \lambda<\lambda_0, \tag{3·5}$$

and finally we obtain

$$W=\bar{H}_0-\sum_{i>k}\sum_{r>s} SS|U_{ik,rs}|^2/[E(i)+E(k)-E(r)-E(s)-2c\gamma\bar{V}]; \tag{3·6}$$

c_W, c_M, c_H, c_B are given by Kemmer[5] and Volz[6] to be $c_W=-1/12$, $c_M=10/12$, $c_H=-2/12$, $c_B=5/12$, from which values we find $c=-9/4<0$. As $\bar{V}>0$ and $E_i+E_k>E_r+E_s$, we find out that the second term of (3·6) is smaller in absolute magnitude than the second order energy of the usual perturbation theory.

Conclusion

A method of solution is given for the system of interacting Fermi-Dirac particles. The essential assumption of the method is not the weak interaction as in the usual perturbation theory but the largeness of the total number of particles. The energy of the ground state is given by a closely analogous expression as the second order energy of the perturbation theory. The result is applied to the discussion of heavy atomic nuclei.

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The Observability of the C-meson

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From the analysis of the various divergences in the field theory, it becomes clear that the C-meson is satisfactory to help the mass-type divergence. However, the existence of the C-meson has not yet been so decisive that it seems necessary to study the magnitude of the C-meson's effect to the various processes and its observability in the expriments, in order to show more concretely that the divergences occuring in the field theory is partly inseperably combined with the structure of the elementary particles. Expecting that the mass of the C-meson is large, the detectable C-meson's effects to the electron problems are hardly obtained. This is because that the range of the C-field is very small compared with the Compton wave length of the electron.[1] But the C-meson's effects to the charged mesons and protons are not surely small. In order to study how much the C-meson effects to such problems, we calculate the process such that the charged spinor particle emits the C-meson being scattered in the Coulomb field of the nucleus, and discuss its influence on the absorption of cosmic-ray underground.

§ 1. The Brems-C-Strahlung

Let $E_0\,\boldsymbol{P}_0$ and $E\,\boldsymbol{P}$ denote the energy and the momentum of the charged spinor particle before and after the Brems-C-Strahlung process respectively. $\boldsymbol{k}$ is the momentum of the C-meson emitted.

Then the differential cross-section of the Brems-C-Strahlung is

$$\begin{aligned}
d\phi=&\frac{1}{2\pi}\,(z^2e^4f^2)\,\frac{p}{p_0}\,\frac{k^2}{k_0}\,dk\,d\Omega_{\boldsymbol{P}}\,d\Omega_{\boldsymbol{k}}\,\frac{1}{q^4}\times\Big[\frac{1}{(x^2-2E_0k_0+2\,(\boldsymbol{P}_0\boldsymbol{k}))^2}\,\{(4\mu^2-x^2)\times\\
&(\boldsymbol{P}_0\boldsymbol{P}-\boldsymbol{P}\boldsymbol{k}+\boldsymbol{P}_0\boldsymbol{k})-(x^2-2E_0k_0+2\,(\boldsymbol{P}_0\boldsymbol{k}))\,(\boldsymbol{P}\boldsymbol{k})+(x^2-2Ek_0)\,(\boldsymbol{P}_0\boldsymbol{k})+E_0E(P_0^2+k^2)\\
&-2E^2P_0^2+3\,(E_0E+E^2-E_0^2)\,\mu^2+4\mu^4-\mu^2k^2+E_0E^3\}+\frac{1}{(x^2+2Ek_0-2\,(\boldsymbol{P}\boldsymbol{k}))^2}\times\\
&\{(4\mu^2-x^2)\,(\boldsymbol{P}_0\boldsymbol{P}-\boldsymbol{P}\boldsymbol{k}+\boldsymbol{P}_0\boldsymbol{k})+(x^2+2Ek_0-2\,(\boldsymbol{P}\boldsymbol{k}))\,(\boldsymbol{P}_0\boldsymbol{k})-(x^2+2E_0k_0)\,(\boldsymbol{P}\boldsymbol{k})\\
&+EE_0(p^2+k^2)-2E_0^2p^2+3\,(E_0E+E_0^2-E^2)\,\mu^2+4\mu^4-\mu^2k^2+EE_0^3\}+\\
&\frac{2}{(x-2E_0k_0+2\,(\boldsymbol{P}_0\boldsymbol{k}))\,(x^2+2Ek_0-2\,(\boldsymbol{P}\boldsymbol{k}))}\,\{(2k_0^2-x^2)\,(\boldsymbol{P}_0\boldsymbol{P})+2\mu^2(p_0^2+p^2+k^2-q^2)\\
&-2\,(\boldsymbol{P}_0\boldsymbol{P})\,(\boldsymbol{P}\boldsymbol{k})+(4E_0E-2k_0^2+4\mu^2+x^2)\,\mu^2-E_0Ex^2\}\Big]
\end{aligned}\tag{1}$$

μ and $\varkappa$ is the mass of the spinor particle and that of the C-meson respectively. $k_0=\sqrt{\varkappa^2+k^2}$ and $q^2=|\boldsymbol{P}_0-\boldsymbol{P}-\boldsymbol{k}|^2$ is the square of the momentum transmitted to nucleus.

We estimate the effect of the screening. If $1/q$ is over the atomic radius the contribution to the cross section get reduced very rapidly. So that we can set the following condition[2)]

$$1/q_{min} \lesssim a_0 z^{-1/3} \qquad a_0: \text{ Bohr radius} \tag{2}$$

while $q_{min}=(k_0^2\mu^2+E_0E\varkappa^2)/2E_0Ek_0$. Considering that the main contribution to the cross section comes from the part $E_0\sim E\sim k$, the minimum incident energy at which the screening becomes effective is

$$E_0' \simeq a_0 z^{-1/3}(\mu^2+\varkappa^2)=137\cdot z^{-1/3}\left(\frac{\mu^2}{m^2}+\frac{\varkappa^2}{m^2}\right)m$$

$$\sim z^{-1/3}\left(\frac{\mu^2}{m^2}+\frac{\varkappa^2}{m^2}\right)\cdot 10^8\, eV, \quad m: \text{ electron mass} \tag{3}$$

E_0' is given in the Table (1), for various values of masses of charged spinor particles.

E_0': $\times z^{-1/3}\cdot 10^8 eV$	10^4	4.7×10^4	8.2×10^4	3.2×10^6
assuming	$\varkappa=100\ m$	$\mu\gg\varkappa$	$\mu\gg\varkappa$	$\mu\gg\varkappa$
μ: $\times m$	1	217	286	1800

Now, we integrate (1) over the angle,[3)] neglecting the contribution of the order $(\mu/E_0)^2$ and $(\varkappa/E_0)^2$ and set $f^2=2e^2$. It results

$$d\phi=4az^2\left(\frac{e^2}{\mu}\right)^2\frac{E}{E_0}\frac{dk}{k_0}\frac{k^2}{K^2}\left\{\left(\frac{k^2}{K^2}+\frac{k^2}{2EE_0}\right)\log\left(\frac{2k}{\mu}\frac{E_0E}{K^2}\right)\right.$$

$$\left.+\left(\frac{k^2}{K^2}-\frac{1}{2}\right)\frac{k^2}{E_0E}-\frac{7}{6}\frac{k^2}{K^2}-\frac{2}{3}\right\}$$

$$K^2=EE_0\delta^2+k^2 \qquad \delta=\varkappa/\mu \qquad E_0=E+k_0\,. \tag{4}$$

§ 2. Energy loss

The energy loss due to the above process of the charged particle penetrating through the matter is estimated. Let N be the number of atom per unit volume, then the energy loss per unit length comes out to be

$$-(dE/dx)_c=N\int_0^{E_0}k_0\,d\phi\,. \tag{5}$$

Substitute (4) into (5) and calculating the two limiting cases, it results

$$-(dE/dx)_c=2aNz^2\left(\frac{e^2}{\mu}\right)^2\left(\frac{\mu}{x}\right)^2E_0\left(\log\frac{2E_0}{K}+1.7\right)\quad \delta\gg 1 \tag{5a}$$

$$-(dE/dx)_c=\frac{8}{3}aNz^2\left(\frac{e^2}{\mu}\right)^2E_0\left(\log\frac{2E_0}{\mu}+1/8\right)\quad \delta\ll 1 \tag{5b}$$

In the case $\delta\sim 1$, $-(dE/dx)_c$ takes the value between (5a) and (5b) and does not largely deviate from the value (5) in which δ is taken equal to 1.

§ 3. The Influence on the cosmic-ray underground

The most part of the cosmic-ray underground is consisted of the μ-meson. The energy loss of the μ-meson is expressed as

$$-(dE/dx)=a+bE \tag{6}$$

a and b correspond to the energy loss due to the ionization and the radiation like processes respectively. The loss due to the decay is negligible in the underground. The energy E of the μ-meson at the depth x is combined with the energy E_0 at sea level in the following relation.

$$E_0=Ee^{bx}+\frac{a}{b}\,(e^{bx}-1)\,. \tag{7}$$

If the integral spectrum $F(E_0)$ of the μ-meson at sea level is known, it is easy to test the value of a and b comparing $F(E_0)$ in which (7) is substituted, with the experimental curve. Let a and b take the value due to ionization and due to radiation and pair creation respectively, and $F(E_0)$ is in a good agreement with the experimental absorption curve underground.[4] If the effects of the C-meson were same order as that of radiation, the absorption curve would go down more rapidly. To avoid this difficulty it is necessary to take the larger value of x the mass of the C-meson. Assuming the effect of the C-meson to be about 1/10 of that of the ordinary radiation of μ-meson, we obtain as the mass of the C-meson

$$x_{\min}\sim 500\,m. \tag{8}$$

If the experimental results of the cosmic-ray is gained in the deeper underground ($\gtrsim 2000\,mH_2O$), the value of x may be determined more precisely. The discussion may be valid it the μ-meson is of scalar type.

§ 4. Other effects

If we take the mass of the C-meson to be order of $500\,m$, it would be expected that the C-meson give the considerable effect to the phenomena concerning to protons. Though an objection against the C-meson hypothesis has been proposed in the problem of the mirror nuclei,[5] it seems not so decisive because in such a problem the considerable effect due to the complicated structure of the nucleus may be expected besides that of the C-meson. But the detectability of the C-meson in the nuclear phenomena may be much reduced by the existence of the neutral nuclear meson. Remarking, however, the difference of the decay process between the C-meson and the neutral nuclear meson, —— that is, the C-meson decays into a electron and a positron while the neutral nuclear meson disintegrates into two or three photons——, the test of the C-meson can be expected in the mixed shower or in the other nuclear collision phenomena.

In conclusion I wish to express my deep gratitude to Professor S. Sakata for his kind advice and encouragement for this work, and I also express my cordial thanks to Mr. Hayakawa for his valuable discussion.

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Note on Very Large Cosmic-Ray Stars

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1. Introduction.

It is well known that the cosmic-ray stars in photographic plates are the high energy nuclear reactions, which are well explained by the evaporation theory of nuclear physics.[1 2] Nevertheless, the large stars show some abnormal phenomena which seemingly contradict with the above theory. According to Perkins et al,[2] α-particles emitted in large stars show remarkable angular collimation and have the energy distribution different from that expected from the Weisskopf's evaporation theory. These authors pointed out that there were two possible explanations for them.

1) Local heating hypothesis ; Before the whole nucleus gets thermal equilibrium, highly heated part of the nucleus emits several α-particles ;

2) Fission hypothesis ; The highly excited nucleus undergoes the fission process, then the smaller fragment disintegrates spontaneously into several α-particles.

Discussing these two possible explanations from various point of view, we are lead to the conclusion that the latter is plausible. Furthermore, the occurrence of the heavy nuclear splinters in large stars seems to support this fission hypothesis, as stated by Perkins et al.[3)4)]

2. The local heating hypothesis.

Even in very large stars, the behaviour of emitted protons does not contradict with the evaporation theory. Thus, the whole nucleus seems to reach the thermal equilibrium in the very beginning of the reaction, and the local heating process will not play a great role.

Theoretically, Tomonaga calculated the viscosity and the thermal conductivity of nuclear matter, and showed these quantities were very large even in highly excited states.[5] Using his results, we can compare the rate of heat conduction and the local evaporation as follows. Let us separate the nucleus into two parts, N_1, N_2 (Fig. 1), and suppose the temperatures of each part at time $t=0$ to be T MeV and 0 MeV respectively. The hot part N_1 loses its excitation energy through the evaporation of particles and the heat conduction to N_2. We write the rates of these energy losses per unit time at the beginning as X_{evap} and X_{th} respectively. Then their ratio is:

$$\frac{X_{\text{evap}}}{X_{\text{th}}} \approx \frac{T^3}{5000}, \tag{1}$$

which is smaller than unity even for $T=10$ MeV. We see, that most of the locally concentrated heat will rapidly diffuse through the whole nucleus.

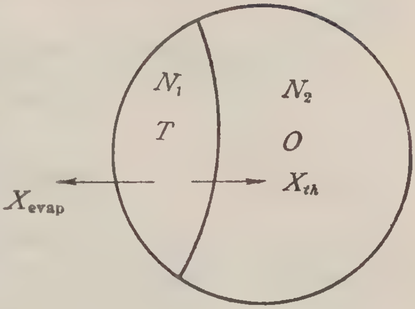

Fig. 1 Local heating.

3. The fission width.

The above estimation shows not only that the local heating process plays little role, but also that the thermal equilibrium is realized also in highly excited state. This will allow us to use Bohr's general idea of nuclear reactions for such a large star. Thus we shall discuss the fission hypothesis extrapolating the Bohr-Wheeler's theory.[6)]

According to their calculation, the fission width Γ_f for a nucleus with mass number A and the excitation energy X is given by

$$\Gamma_f(X) = \frac{1}{2\pi\rho_A(X)} \int_{E_f}^{X} \rho_A^*(X-\varepsilon)\, d\varepsilon, \tag{2}$$

where E_f is the fission threshold energy, ρ_A is the level density of the mother nucleus, and ρ_A^* is the level density at the saddle point of the potential surface in a fission process.

We assume $\rho_A^* = \rho_A$, and adopt the Fermi gas model for a nucleus, which is experimentally confirmed by Perkins et al.[2)] Then we can give the explicit form of the fission width, that is

$$\Gamma_f(X) \approx \frac{T}{2\pi}\, e^{-E_f/T}. \tag{3}$$

T means the nuclear temperature,

$$T \approx 3.17\sqrt{X/A} \tag{4}$$

On the other hand, the neutron width Γ_n is in the same appoximation,

$$\Gamma_n(X) \approx \frac{1}{2\pi} \frac{A^{2/3}}{K'}\, T^2 e^{-E_b/T}, \tag{5}$$

$$K' = \hbar^2/2mr_0^2 \sim 10 \text{ MeV},$$

where $E_b \fallingdotseq 8$ MeV is the binding energy of a neutron. Comparing Γ_f with Γ_n, we see that the fission process is rare, i.e. $\Gamma_f \ll \Gamma_n$, as long as $E_f \gg E_b$. The threshold E_f must be nearly equal to E_b for the fission process to occur frequently.

According to Bohr-Wheeler, the fission threshold energy E_f is given by

$$E_f \doteq 4\pi r_0^2 O A^{2/3} f(x), \tag{6}$$

$$x = \frac{Z^2/A}{(Z^2/A)_{\rm lim}}.$$

The notations are all the same as that of Bohr-Wheeler. They gave the following numerical constants from the data of stable nuclei.

$$\left.\begin{aligned} 4\pi r_0^2 O &= 14\ MeV, \\ (Z^2/A)_{\rm lim} &= 47.8. \end{aligned}\right\} \tag{7}$$

From (6) and (7), we get for the Ag-nucleus,

$$E_f \sim 45\ {\rm MeV}. \tag{8}$$

This value is much larger than $E_b \fallingdotseq 8$ MeV. It corresponds to the fact that the Ag-nucleus does not undergo fission in ordinary (low energy) nuclear reactions.

If the fission hypothesis is true, however, the threshold energy E_f for highly excited Ag-nucleus should be much smaller than (8). Now we are left the following two possibilities for this to be the case; i.e.,

i) increase of the Coulomb repulsions,
ii) decrease of the surface tension of the nucleus.

with the increasing excitation energy.

The highly excited heavy nucleus evaporates mostly neutrons and protons are prevented by the Coulomb potential. This may result in the relative increase of the Coulomb repulsion, i.e. i), and make the threshold energy E_f lower. In fact, the fission of Bi by the fast neutron bombardment are known to be due to this effect.7) For Ag or Br-nucleus, however, this effect is too small; e.g. only after the evaporation of forty neutrons and no proton the threshold energy becomes ~ 6 Mev. Thus we incline to take the hypothesis ii).

4. The change of surface tension.

Here, we discuss the last possibility, i.e. the decease of the surface tension with the increasing excitation energy, adopting various models for nuclei. Let

$$O(T) = 14\ {\rm MeV} + O'(T) \quad (O'(0)=0) \tag{9}$$

(Note: area is measured in units of $4\pi r_0^2$.)

be the surface tension of the nucleus at the temperature T. The total surface energy of the nucleus is denoted by $U(T)A^{2/3}$. These two quantities are connected by the well-known thermodynamical relations:

$$U = O' - T\frac{\partial O'}{\partial T}. \tag{10}$$

If we adopt the Fermi gas model for the nucleus, U is given by

$$\begin{aligned} UA^{2/3} &\approx (\textit{Total excitation energy}) \\ &\times \frac{(\textit{Nuclear surface})(\textit{Nucleon radius})}{(\textit{Nuclear volume})} \\ &\approx 0.175\,A^{2/3}\,T^2. \end{aligned} \tag{11}$$

The solution of (10) with (11) is

$$O' = (\textit{constant})\ T - U, \tag{12}$$

where the integral constant is determined by the condition that the surface tension $O = 14 + O'$ must vanish at the critical temperature T_c, which may be tentatively determined as follows.

$$\frac{A}{10}\,T_c^2 = \textit{total binding energy} \approx 8\,A\ \text{Mev}. \tag{13}$$

This results in ≈ 9 MeV for T_c. Thus we have the following temperature dependence of the surface tension.

$$O = 14\,\frac{T_c^2 - T^2}{T_c^2}.\quad (\text{Mev}). \tag{14}$$

While the classical formula of Eötvös shows the linear temperature dependence:

$$O = 14\,\frac{T_c - T}{T_c}\quad (\text{Mev}). \tag{15}$$

These two expressions, (14) and (15), correspond to two extreme cases. and other models, including the liquid drop model*), lie between them. We calculate

Table I The fission width for the Ag nucleus,

X (MeV)	T (MeV)	Γ_n (MeV)	Γ_f (MeV)		Ratio Γ_f/Γ_n	
			1*)	2**)	1*)	2**)
11	1	1.2×10^{-4}	1.5×10^{-14}	6.7×10^{-19}	0	0
44	2	2.6×10^{-2}	3.9×10^{-5}	4.8×10^{-9}	0	0
98	3	0.22	2.4×10^{-2}	8.2×10^{-6}	0.10	0
173	4	0.76	0.39	9.6×10^{-4}	0.34	0
270	5	0.89	0.8	3.3×10^{-2}	0.42	0.036
389	6	3.3	–	0.48	–	0.13
530	7	5.3	–	1.12	–	0.17

Note *) 1: linear case
**) 2: quadratic case

the fission threshold energy E_f using the above relations and then the width and show their numerical results in Table I. and Fig. 2.

5. Comparison with experiments.

From Table I, we see Γ_f becomes the same order as Γ_n at $T\approx3$ MeV in the classical case and the fission process may be possible at $T\gtrsim 3$ MeV ($X\gtrsim 100$ MeV for $A\approx 100$). While, the fission will not occur up to $T\approx 6$ MeV ($X\approx 40$ MeV for $A=100$) in the quadratic case.

According to Perkins, only in large stars with ten or more prongs the energy distribution of emitted α-particles is different from that expected from the evaporation theory and their angular distribution is not uniform but shows collimation. This leads Perkins *et al* to the view that a nucleus undergoes fission in the first stage and then the smaller fragment spontaneously disintegrates into several α-particles, so that these α-particles are collimated preserving the momentum of the mother fragment. On the other hand, the larger fragment undergoes evaporation as usual. Furthermore there are direct evidences of fission that one observes one or two heavy splinters in a star. These experiments may be favourable to our theory.

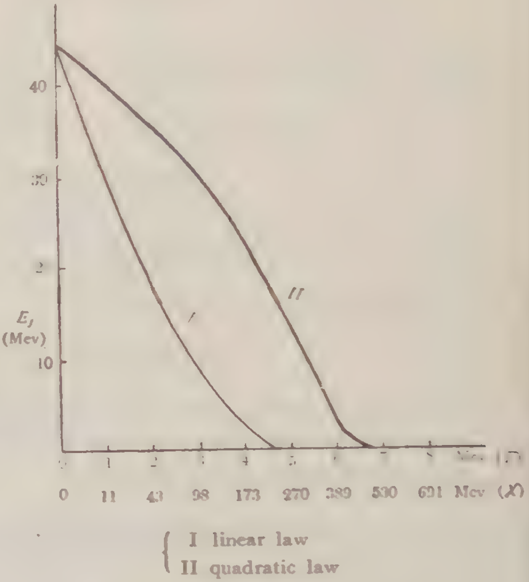

Fig. 2.

Change of the fission threshold energy E_f with the temperature.

But such a fission-like process is not observed in photographic experiments of Berkeley (100 MeV neutron and 200 MeV deuteron bombardment). This indicates the small fission probability of the Ag and Br nucleus with the excitation energy 100～200 MeV and will favour to the quadratic law of the surface tension-temperature relation. Thus our calculation may justify the fission hypothesis in the large stars, leaving unsolved the problem of the asymmetry of fission. (*to be continued*)

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*) In the case of the liquid drop model, the relation of surface tension to temperature resembles the quadratic law (14). It seems necessary to assume that the total excitation energy of the nucleus consists of only the surface energy in the liquid drop model. This fact has already been pointed out by Perkins et. al.[2) from their experiments. Detailed account will be given in the subsequent paper.

Note added in proof.

Hereto we considered the "quasi-static", i.e., Bohr-Wheeler's mechanism of the fission process. But the large asymmetry in fission fragments will not be explained by the above quasi-static mechanism. It seems rather plausible that these fission-like processes might be caused by the new dynamical mechanism because of very high excitation of the nuclei. Our estimation of the fission width is certainly the lower limit of true one. Thus we may conclude that the above discussion of fissionability would be valid even in the dynamical fission process.

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The Use of the Proper Time in Quantum Electrodynamics I.

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I. Introduction

The space-time approach to quantum electrodynamics, as has been developed by Feynman,[1] seems to offer a very attractive and useful idea to this domain of physics. His ingenious method is indeed attractive, not only because of its intuitive procedure which enables one to picture to oneself the complicated interactions of elementary particles, its ease and relativistic correctness with which one can calculate the necessary matrix elements or transition probabilities, but also because of its way of thinking which seems somewhat strange at first look and resists our minds that are accustomed to causal laws. According to the new standpoint, one looks upon the world in its four-dimensional entirety. A phenomenon that will come into play in this theatre is now laid out beforehand in full detail from immemorial past to ultimate future and one investigates the whole of it at glance. The time itself loses sense as the indicator of the development of phenomena; there are particles which flow down as well as up the stream of time; the eventual creation and annihilation of pairs that may occur now and then, is no creation nor annihilation, but only a change of directions of moving particles, from past to future, or from future to past; a virtual pair, which, according to the ordinary view, is foredoomed to exist only for a limited interval of time, may also be regarded as a single particle that is circulating round a closed orbit in the four-dimensional theatre; a real particle is then a particle whose orbit is not closed but reaches to infinity. . .

In such a view, a state with prescribed number of particles including real as well as virtual does not exactly correspond to a four-dimensional state in the ordinary sense, that is, a state represented by the wave function satisfying the time dependent Schroedinger equation. But the former is rather a part of the latter in which any number of virtual particles may be allowed to occur. To obtain an idea of the actual state we shall have to sum over all possibilities as to the number of virtual particles.

The interpretation of the four-dimensional state in the present sense becomes also somewhat different from the conventional one as giving the transition probability or amplitude from a given state A to a final state F in the three-dimensional space. We can rather ask for the relative probability that a four-dimensional state A-I-F with prescribed real as well as virtual (intermediate) particles be realized in nature. This will be zero unless the arbitrary chosen A-I-F is not such as is an actually possible transition under the Schroedinger equation.

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The above-mentioned view of the entire space-time behavior of nature *sub specie aeternitatis*, however, might not appeal to a reason which is liable to think in the language of differential equations and pursue the development of things along a certain parameter. In fact we find it hard to regard the world line of a particle as a mere status of that particle, but are unconciously following the motion of an imaginary mass point along the world line. Thus, in Feynman's theory where the ordinary time loses its rôle as the indicator of the development of the world, it would still be convenient to introduce some parameter with which the four-dimensional world is going to shape itself. How this is possible to a certain extent we shall see in what follows.

2. Formal introduction of the proper time

Let us consider a wave function obeying the ordinary Schroedinger equation

$$i\frac{\partial}{\partial t}\psi(t,x)=H(x)\,\psi(t,x). \tag{1}$$

The scalar product (ψ, φ) of two wave functions ψ and φ at a given time carries the meaning of the probability amplitude for finding a state characterized by ψ when we know that the system is in the state φ. This interpretation is based upon the mathematical fact that the length of the wave function vector is constant in time according to (1). If we go over to the four-dimensional standpoint and regard the behavior of ψ in both t and x for a finite interval of time as characterizing the state $\Psi(t, x)$, we shall naturally have to define the norm of a wave function by

$$(\Psi, \Psi)=\int_{t_0}^{t_0+T}\int_{-\infty}^{\infty}(\psi(tx), \psi(tx))\,dx\,dt=T\int(\psi(x), \psi(x))\,dx, \tag{2}$$

which is a multiple of the ordinary norm of the state ψ. Thus we have only to alter the normalization of the wave function for the transition from ψ to Ψ, though this means an infinite factor when the time interval is extended indefinitely. The probability amplitude (ψ, φ) mentioned above is then in the new standpoint merely a probability amplitude density for a cross section of time from which the full amplitude is derived by integration with respect to t:

$$(\Psi, \Phi)=\int(\psi, \phi)\,dt. \tag{3}$$

Eq. (3) allows the following interpretation. Suppose we take for Ψ and Φ two stationary states for a system, then (3) will be zero unless Ψ and Φ represents the same state. But if some perturbation is introduced in the system, the system will no longer remain in the original (three-dimensional) state φ, but the space-time behavior of Φ will be expressed by decomposing it with respect to unperturbed eigenfunctions,

$$\Phi=\sum_k a_k(t)\,\Phi_k(t), \qquad a_k(t_0)=\delta_{k0}. \tag{4}$$

The probability that we find the system after an infinite lapse of time in the state k is given by

$$\lim_{t\to\infty}|a_k(t)|^2=\lim_{t\to\infty}|(\psi_k,\Phi)|^2. \tag{5}$$

If we displace the time scale and shift the initial point to $-\infty$, we obtain the probability for finding a state Ψ_k or ψ'_k (referred to unperturbed coordinates) irrespective of time when we know the system was in the state φ and the perturbation has been, say adiabatically, switched on, by

$$|a_k(\infty)|^2=\left|\int_{-\infty}^{\infty}(\Psi_k(t),\Phi(t))\,dt\right|^2=|(\Psi_k,\Phi)|^2, \tag{6}$$

for we may suppose that $a_k(t)$ has reached its stationary value for any finite t. In this way the four-dimensional scalar product acquires a physical meaning.

Now let us investigate the problem from a different point of view. The Schroedinger equation (1) implies, when regarded four-dimensionally, a sort of supplementary condition imposed on Ψ:

$$(i\partial/\partial t-H)\Psi=0 \tag{7}$$

since we no longer look upon t as a parameter along which Ψ develops itself. Consequently a Ψ which corresponds to the real world must be of the form

$$\Psi_{\text{real}}=\delta\left(i\frac{\partial}{\partial t}-H\right)\Psi_0. \tag{8}$$

If we introduce here a redundant variable τ and assume an equation of the type

$$i\frac{\partial}{\partial\tau}\Psi=\left(i\frac{\partial}{\partial t}-H\right), \tag{9}$$

and an accompanying eigenvalue problem

$$\lambda\Psi=\left(i\frac{\partial}{\partial t}-H\right)\Psi, \tag{9'}$$

(8) becomes

$$\Psi_{\text{real}}=\delta(\lambda)\,\Psi_0=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Psi(\tau)\,d\tau. \tag{10}$$

The four-dimensional view has a static character in that a state is defined once for all t and after that a condition is invoked in order that it correspond to any reality. But now that τ is introduced and (9) has regained the aspect of an ordinary Schroedinger equation, the new variable must play a similar rôle as that the ordinary time played in (1). As τ goes on, a wave packet localized over certain four-dimensional volume will move to and fro and change its shape gradually. Then we may think of τ as something like the proper time of the particle represented by the wave packet. If we directed a camera on to that particle with the shutter open for an infinitely long time, we should obtain a vague strip of world line as the locus of the particle, which would correspond to the real wave function Ψ_{real}. Eq. (10) tells us just this situation in the mathematical language.

Next we shall consider the transition probability. If a perturbation term is inserted, (9) becomes

$$i\frac{\partial}{\partial t}\Psi=\left(i\frac{\partial}{\partial t}-H_0-H_1\right)\Psi, \tag{11}$$

while the free particle wave functions to which we refer the initial and final state satisfy the equation

$$\left(i\frac{\partial}{\partial t}-H_0\right)\Psi_A=\left(i\frac{\partial}{\partial t}-H_0\right)\Psi_F=0. \tag{12}$$

Let us write $i\partial/\partial t-H_0\equiv L$, and perform the transformation

$$\Psi=\exp\left[-iL\tau\right]\Psi_1.\quad H_1'=\exp\left[iL\tau\right]H_1\exp\left[-iL\tau\right], \tag{13}$$

(11) then goes over into

$$i\frac{\partial}{\partial t}\Psi_1=-H_1'\Psi_1. \tag{14}$$

Starting at $\tau=\tau_0$ from a free state Ψ_A, the transition amplitude at $\tau=\tau$ to a free state Ψ_F will be expressed by

$$P(AF)=\delta(L)(F|U(\tau\tau_0)|A),\quad \Psi(\tau)=U(\tau\tau_0)\Psi(\tau_0), \tag{15}$$

since H_1' would in general bring a real particle into unreal one for which $L\Psi\neq0$. Eq. (15) admits a twofold interpretation: a) formally, $P(AF)$ is given from $(\Psi_F, \Psi(\tau))$ by taking the zero-frequency component of $\Psi(\tau)$:

$$P(AF)=\frac{1}{2\pi}\int_{-\infty}^{\infty}(\Psi_F, \Psi(\tau))\,d\tau\,; \tag{16}$$

b) physically, it is the accumulated amplitude for the transition starting from the state A and arriving at F after an infinite lapse of time:

$$P(AF)=\int_{-\infty}^{\infty}(\Psi_F, \Psi(\tau))\,d\tau=\int_{-\infty}^{\infty}(F|U(\tau,-\infty)|A)\,d\tau. \tag{17}$$

(16) and (17) differ only by a normalization factor.

Thus we see that the formal introduction of a redundant parameter and its identification with the proper time carries a bit of mathematical convenience as well as physical plausibility. In order to convince ourselves further on this point, we shall next recall Fock's theory and go ahead on his line.

3. Theory of Fock and its extension

Fock[2)] once introduced the concept of the proper time in the Dirac electron in parallel with the classical theory and proved its correspondence to the ordinary proper time. It will be briefly recapitulated below. The classical Lagrangian for an electron interacting with the electromagnetic field is given by

$$L_0 = -mc\dot{x}_\mu^2 - mc^2/2 - (e/c)(\dot{x}_\mu A_\mu), \quad \dot{x}_\mu = dx_\mu/d\tau, \tag{18}$$

and the equation of motion follows from the variational principle

$$\delta S = 0, \quad S = \int^\tau L_0 d\tau, \tag{19}$$

together with the supplementary condition

$$\dot{x}_\mu^2 = \text{const.} = -c^2. \tag{20}$$

Eliminating τ from (19) and (20) we get the ordinary action function $S(x_\mu)$ for the system. The Hamilton-Jacobi partial differential equation which follows from above is

$$\frac{\partial S}{\partial \tau} + \frac{1}{2m}\left[\left(\text{grad } S + \frac{e}{c} A\right)^2 - \frac{1}{c}\left(\frac{\partial S}{\partial t} - e\Phi\right)^2\right] + m^2c^2 = 0, \tag{21}$$

with the condition

$$\partial S/\partial \tau = 0. \tag{21'}$$

Now turning to quantum theory, the Dirac electron obeys the wave equation

$$(\gamma_\mu D_\mu + \varkappa)\psi = 0, \quad D_\mu = \partial/\partial x_\mu - e/\hbar c \cdot A_\mu. \tag{22}$$

Such a ψ may be expressed as

$$\psi = (\gamma_\mu D_\mu - \varkappa)\Psi, \tag{23}$$

where Ψ in turn shall obey the second order differential equation

$$(\gamma D + \varkappa)(\gamma D - \varkappa)\Psi = \left[\left(\Box - \frac{ei}{\hbar c} A\right)^2 - \varkappa^2 + \frac{e}{\hbar c} \sigma_{\mu\nu} F_{\mu\nu}\right]\Psi \equiv \Lambda\Psi = 0. \tag{24}$$

Fock proposed to introduce a function F which satisfies instead of (24) an equation closely analogous to (21):

$$i\hbar \frac{\partial}{\partial \tau} F = \frac{\hbar^2}{2mc} \Lambda F, \tag{25}$$

and express Ψ as a suitable integral of F over τ:

$$\Psi = \int_C F d\tau, \tag{26}$$

the contour being taken in such a way that

$$\frac{\partial}{\partial\tau}\Psi = \int_C \frac{\partial F}{\partial\tau} d\tau = F|_C = 0, \tag{27}$$

which corresponds to the condition (21′).

When we neglect temporarily the electromagnetic interaction, the electron becomes free and (25) is solved by

$$F(\tau) = \exp\left(-\frac{i\hbar\tau}{2mc}\Lambda\right) F(0). \tag{28}$$

Given an initial distribution of the wave packet over space-time, it will change with τ by the relation (28). But only those stationary states with zero eigenvalue can ever correspond to a real world:

$$\Lambda F = \lambda F = 0. \tag{29}$$

In classical language, a preassigned pattern of streamlines representing the motion of an aggregate of electrons will change according to the dynamical law (21). Only the stationary stream is the true state of the world where individual electrons follow invariant paths.

Now let us determine the behavior of the wave packet which starts from a delta function $F(0) = \delta(x)$ at $\tau = 0$. The answer is easily given by the Fourier representation

$$\begin{aligned} F(\tau, x) &= \left(\frac{1}{2\pi}\right)^4 \int e^{i\lambda\tau} e^{ik_\mu x_\mu}\, \delta(\lambda + (k^2 + \varkappa^2)/2\varkappa)\, dk_\mu d\lambda \\ &= \frac{\varkappa}{\pi} \int \Delta^{(1)}(x^2, \varkappa^2 + 2\varkappa\lambda)\, e^{i\lambda\tau} d\tau \equiv \varkappa \Delta_\varkappa(\tau, x^2) \\ &= -\frac{i}{4\pi^2}\frac{\varkappa^2}{\tau^2} \exp\left(\frac{\varkappa}{2\tau} x^2 i + \frac{\varkappa}{2}\tau i\right), \qquad (\tau > 0). \end{aligned} \tag{30}$$

$F(\tau, x)$ is the probability amplitude for finding the particle at x when we know that it was at the origin a time τ ago. According to the previous argument the real observed amplitude will be obtained when we sum $F(\tau, x)$ for all $\tau > 0$, which results in

$$\begin{aligned} \int_0^\infty F(\tau, x)\, d\tau &= \frac{2\varkappa}{(2\pi)^3} \int_0^\infty \delta_+(k^2 + \varkappa^2)\, e^{ik_\mu x_\mu}\, dk_\mu \\ &= \frac{2\varkappa}{i}\left(\bar{\Delta}(x) + \frac{i}{2}\Delta^{(1)}(x)\right) = \varkappa \Delta_F. \end{aligned} \tag{31}$$

Here $\bar{\varDelta}$, $\varDelta^{(1)}$, and $\varDelta_F$ bear the meaning as defined in Schwinger,[3] Feynman,[1] and Dyson[4] :

$$\bar{\varDelta}(x)=\frac{1}{(2\pi)^4}\int\frac{1}{k^2+\varkappa^2}\,e^{ik_\mu x_\mu}\,dk_\mu,$$

$$\varDelta^{(1)}(x)=\frac{1}{(2\pi)^3}\int\delta(k^2+\varkappa^2)\,e^{ik_\mu x_\mu}\,dk_\mu,$$

$$\varDelta_F(x)=-2i\bar{\varDelta}(x)+\varDelta^{(1)}(x)=\frac{2}{(2\pi)^3}\int\delta_+(k^2+\varkappa^2)\,e^{ik_\mu x_\mu}\,dk_\mu. \tag{32}$$

Thus we see that we shall be able to arrive at $\varDelta_F$, the fundamental quantity in positron theory, if we take $F(\tau, x)$ for the (five-dimensional) commutation relation between quantized wave functions satisfying (25). The proper time τ is nothing but what has been a mere parameter in the integral representation of the $\varDelta$-functions. The integration with respect to τ followed above, not from $-\infty$ to $+\infty$ but only for positive τ, is a departure from the standpoint expounded before, and corresponds to the fact that four-dimensional outgoing waves divergent from a source into past and future are exclusively considered in Feynman's theory.

Now the starting equation (25) seems somewhat artificial and impairing the simplicity of the original Dirac equation. In fact one would be tempted to introduce the proper time as a fifth coordinate in a linearized form. Thus one may put for instance

$$\gamma_5\frac{\partial}{\partial\tau}\,\psi=\left(\gamma_\mu\frac{\partial}{\partial x_\mu}+\varkappa\right)\psi,\quad \gamma_5\frac{\partial}{\partial\tau}\,\psi^\dagger=\left(\gamma_\mu\frac{\partial}{\partial x_\mu}-\varkappa\right)\psi^\dagger, \tag{33}$$

which, by iteration, yields

$$\left(\frac{\partial^2}{\partial\tau^2}+\Box^2-\varkappa^2\right)\psi=\left(\frac{\partial^2}{\partial\tau^2}+\Box^2-\varkappa^2\right)\psi^\dagger=0. \tag{34}$$

A solution of (24) can be written as

$$\psi=\left(\gamma_5\,\frac{\partial}{\partial\tau}+L^\dagger\right)\varPhi,\quad (\Box_\sigma^2-\varkappa^2)\,\varPhi=0,$$

$$L=\gamma_\mu\frac{\partial}{\partial x_\mu}+\varkappa,\quad L^\dagger=\gamma_\mu\frac{\partial}{\partial x_\mu}-\varkappa,\quad \sigma=1,\ldots\ldots5,\ x_5\equiv\tau.. \tag{35}$$

For the commutation relation we adopt the expression

$$\{\psi,\,\psi^\dagger\}=\left(\gamma_5\,\frac{\partial}{\partial\tau}+L^\dagger\right)f(\tau,x),$$

$$f(\tau,x)=-\frac{2i}{(2\pi)^4}\int_0 e^{ik_\sigma x_\sigma}\,\delta(k_\sigma{}^2+\varkappa^2)\,dk_\sigma, \tag{36}$$

where C means the path of integration for k_5: $+\infty i \rightarrow 0 \rightarrow +\infty$. First, when integrated over $\tau = x_5$ from 0 to ∞, (36) gives

$$\int_0^\infty f(\tau, x)\, d\tau = -\frac{2i}{(2\pi)^4}\int_c e^{ik_\mu x_\mu}\, \delta(k_\sigma^2 + x^2)\left[\pi\delta(k_5) - \frac{1}{ik_5}\right] dk_\sigma$$

$$= -\frac{2i}{(2\pi)^4}\int e^{ik_\mu x_\mu}\left[\frac{\pi}{2}\,\delta(k_\mu^2 + x^2) + \frac{1}{2i(k_\mu^2 + x^2)}\right] dk_\mu = -\frac{i}{2}\,\Delta_F(x), \quad (37)$$

and (36) reduces to

$$\int_0^\infty \{\psi(\tau),\ \psi^{\dagger\prime}(0)\}\, d\tau = -\frac{i}{2} L^\dagger \Delta_F(x), \quad (38)$$

the term with $\partial/\partial\tau$ being put to zero on integration. The initial value of f for $\tau = 0$ is calculated in a similar way, and yields the result

$$\left(\frac{\partial}{\partial\tau}\gamma_5 - L^\dagger\right) f\,\Big|_{\tau=0} = \frac{-i}{(2\pi)^4}\int\left(i\gamma_5 - \frac{L^\dagger}{(-k_\mu^2 - x^2)^{1/2}}\right) e^{ik_\mu x_\mu}\, dk_\mu. \quad (39)$$

But the factor in the bracket becomes

$$i\gamma_5 - \frac{L^\dagger}{(-k_\mu^2 - x^2)^{1/2}} = i\gamma_5(1+\varepsilon), \quad \varepsilon = \gamma_5 L^\dagger/(-k_\mu^2 - x^2)^{1/2}, \quad (40)$$

$$\varepsilon^2 = 1, \quad \varepsilon = \pm 1.$$

Thus, (39) is not of the form const. $\delta(x)$, as it should be. This is because we are selecting only those waves which are outgoing from the source point. Neglecting ε, or taking its average, we get the reasonable result

$$<\{\psi(x),\ \gamma_5\psi^\dagger(x')\}>_{\tau=\tau'} \sim \delta(x). \quad (41)$$

In this point the linearized form does not prove to be much preferable to the original Fock equation. Though it may be convenient when we try to make use of the interaction representation, we shall follow hereafter Fock's procedure which seems most natural after all.

4. Problem of vacuum polarization

Now we shall turn to the interpretation of the various terms of the S matrix. The starting point is that the probability amplitude for an electron going from x to x' in a time τ is assumed to be given by

$$(\Psi^*,\ \psi(x, \tau)) = \left(\gamma_\mu \frac{\partial}{\partial x_\mu} - \kappa\right)\Delta(\tau, x - x') \equiv S(\tau, x - x'). \quad (42)$$

If, for example, we consider the self-energy of the electron, we have to do with the process: an electron and a photon start simultaneously at x and afterwards meet again at x', thereby giving rise to the matrix element

$$-\frac{e^2}{\hbar c}\gamma_\mu S(\tau, x-x')\gamma_\mu D_1(x-x'). \tag{43}$$

Integrating over τ we get the usual self-energy element. We could also introduce another proper time for the photon at least formally, modifying $D(x-x')$ to $D(\tau, x-x')$. But then the radiation field no more remains real.

The problem of the vacuum polarization is a more interesting subject. Here we shall confine ourselves to the external polarization only. An electron, starting from x, suffers a scattering by the field at x' (time τ), then comes back to the origin (time $\tau+\tau'$) and there produces a polarization current δj_μ. This will be given by

$$\delta j_\mu = i\frac{e}{\hbar c}\, Tr[\gamma_\mu S(\tau', x-x')\, A_\nu\gamma_\nu S(\tau, x'-x)]. \tag{44}$$

Integration with respect to τ and τ' leads to the usual exppession. If we consider the possibility that an electron can be scattered by the field over and over again before return, then we are dealing with an electron moving under continuous influence of the field. Such an electron will be described by the wave function satisfying

$$\lambda\Psi_\lambda = \Lambda\Psi_\lambda. \tag{45}$$

Its contribution to the induced current is

$$i\int_0^\infty (\Psi^*(x)\, e^{-i\lambda\tau}\gamma_\mu\Psi_\lambda(x))\, d\tau = 2\pi i\delta_+(\lambda)\,\Psi_\lambda^*(x)\,\gamma_\mu(\gamma_\nu D_\nu-\varkappa)\,\Psi_\lambda(x). \tag{46}$$

Summing over all stationary states (and adjusting the normalization factor), we get the induced current expression

$$\delta j_\mu = i\sum_\lambda \delta_1(\lambda)\,\Psi_\lambda^*\gamma_\mu(\gamma_\nu D_\nu-\varkappa)\,\Psi_\lambda. \tag{47}$$

Letting the external field vanish, (47) reduces simply to

$$\delta j_\mu^0 = i\,\mathrm{Tr}\left[\gamma_\mu\left(\gamma_\lambda\frac{\partial}{\partial x_\lambda}-\varkappa\right)\frac{1}{(2\pi)^4}\int\delta_+(k^2+\varkappa^2)\,e^{ik_\mu x_\mu}\,d^4k_\mu\right]_{x=0} = \frac{i}{2}\,\mathrm{Tr}\,[\gamma_\mu S_1(0)], \tag{48}$$

which may be regarded as zero by virtue of symmetry. If, however, (47) is expanded in powers of the coupling constant, an expression of the form

$$\delta j_\mu = c_1 A_\mu + c_2\Box^2 A_\mu + \cdots\cdots \tag{49}$$

will be obtained. The first and second term are divergent, corresponding to the self-energy of photon and the renormalization of the external charge respectively. We shall show, however, that there is a different method of approach to determine the form of δj_μ.

Fock resorted to a kind of the W.K.B. method to solve the proper time wave equation (25), setting

$$F = e^{\frac{i}{\hbar}S_f}, \tag{50}$$

where S is the classical action function satisfying (21). This method is applied to obtain the Riemann function R for the Dirac electron, which is expressed as

$$R=\oint F d\tau, \tag{51}$$

the contour encircling the origin $\tau=0$ in the complex domain. F can be rigorously solved when the electron is free or at least the external field is constant. In the former case, in particular,

$$S=+\frac{m}{2\tau}x^2-\frac{1}{2}mc^2\tau, \qquad f=-\frac{m}{8\pi^2\hbar c}\frac{1}{\tau^2},$$

$$R=-\frac{m}{8\pi^2\hbar c}\oint e^{\frac{im}{2\hbar\tau}x^2-\frac{imc^2}{2\hbar}\tau}\frac{d\tau}{\tau^2}=-\frac{m}{4\pi\hbar\sqrt{-x^2}}J_1\left(\frac{mc}{\hbar}\sqrt{-x^2}\right). \tag{52}$$

On the other hand, Schwinger's $\bar{\Delta}$-function has the integral representation

$$\bar{\Delta}(x)=\frac{1}{8\pi^2}\int_{-\infty}^{\infty}e^{-i\alpha x^2+i\frac{\varkappa^2}{4\alpha}}d\alpha=\frac{\varkappa}{16\pi^2}\int_{-\infty}^{\infty}e^{-\frac{ix^2}{2\tau}\varkappa+\frac{i\varkappa\tau}{2}}\frac{d\tau}{\tau^2}. \tag{53}$$

When $-x^2$ is positive, we may take a contour integral going round the upper half plane at infinity and deviating slightly below the real axis near the origin; when $-x^2$ is negative, we go round the lower half plane at infinity. Then the former contour shrinks to a circle around zero, giving just (43), and the latter integral vanishes. Thus

$$\bar{\Delta}(x)=-\frac{1}{2}R(x)\gamma(-x^2), \tag{54}$$

where

$$\gamma(\lambda)=\begin{cases}1, & \lambda>0\\ 0, & \lambda<0\end{cases}$$

From this result we see that the function $\Delta_F^{(e)}$ in the presence of the external field will be obtained if we choose in (42) a straight integration path from 0 to ∞ instead of the circle around zero. When there is a constant magnetic and electric field H and E parallel to the z axis, $F=\exp(iS/\hbar)f$ is given by

$$S=S_0-\frac{1}{2}mc^2\tau+\frac{eE}{4c}\left[(z-z')^2-c^2(t-t')^2\right]\operatorname{cth}\frac{eE\tau}{2mc}$$
$$+\frac{eH}{4c}\left[(x-x')^2+(y-y')^2\right]\operatorname{ctg}\frac{eH\tau}{2mc},$$

$$S_0=\frac{1}{2}eE(z+z')(t-t')+\frac{eH}{2c}(x'y-y'x), \tag{55}$$

$$f=\frac{m}{8\pi^2\hbar c}\frac{eH}{2mc}\frac{eE}{2mc}f_0\Big/\sin\frac{cH\tau}{2mc}\operatorname{sh}\frac{eE\tau}{2mc},$$

$$f_0=\exp\left[\frac{ie}{2mc}\sigma_z H\tau+\frac{e}{2mc}\alpha_z E\tau\right],$$

and

$$-\frac{1}{2}\Delta_F^{(e)}(x,x')=\int_0^\infty F(x,x')\,d\tau. \tag{56}$$

The polarization current is then

$$\delta j_\mu=i\,Tr\left[\gamma_\mu\left(\gamma_\lambda\frac{\partial}{\partial x_\lambda}-\kappa-\frac{ei}{\hbar c}\gamma_\lambda A_\lambda\right)\Delta_F^{(e)}(x,x')\right]_{x=x'}, \tag{57}$$

where

$$A_x=-\frac{1}{2}Hy,\quad A_y=\frac{1}{2}Hx,\quad A_z=0,\quad A_4=Ezi. \tag{58}$$

Let us consider the problem of the gauge invariance. When A_μ is replaced by $A_\mu+\partial\chi/\partial x_\mu$, S acquires an additional term $(e/c)(\chi-\chi')$ so that $\Delta_F^{(e)}$ is multiplied by a factor $\exp[i(\chi(x)-\chi(x')/\hbar]$. But the current (48) turns out to suffer no modification:

$$\delta j_\mu'=e^{i(\chi-\chi')/\hbar}\,\delta j_\mu|_{x=x'}=\delta j_\mu. \tag{59}$$

The gauge invariance is thus guaranteed. The charge conservation law also holds since

$$\left(\gamma\Box+\kappa-\frac{ei}{\hbar c}A\gamma\right)S_F^{(e)}=S_F^{(e)}\left(-\gamma\Box'+\kappa-\frac{ie}{\hbar c}A'\gamma\right)=-\delta(x-x'). \tag{60}$$

These relations are a consequence of the equation (25) for F and the initial condition: $F(x)=\delta(x)$ at $\tau=0$.

On the other hand the usual perturbation formula (40) gives in general non-gauge invariant, divergent results when evaluated numerically. In our expression (56) for the particular constant field, however, δj_μ turns out to vanish if we regard expressions of the type $(x_\mu-x_\mu')\Delta_F^{(e)}(x,x')|_{x=x'}$ to be zero. This may be approved because here appears only one singular function and not a product of two or more, so that it is a consistent condition compatible with other requirements. It is also shown in the appendix that no essential difficulties arise in case of an arbitrary constant field. The self-energy of the photon is then zero at least in the order e^2, for the first term in the expansion (40) vanishes according to the above result.

Appendix. Solution for an arbitrary constant external field.[5]

We take the constant external field $F_{\lambda\mu}=-F_{\mu\lambda}$, and the vector potential for it:

$$A_\mu=\frac{1}{2}(x_\lambda-x_{0\lambda})F_{\lambda\mu}, \tag{A1}$$

where $x_{0\lambda}$ are arbitrary constants corresponding to an initial condition in the subsequent calculation. The classical equation of motion is then

$$\frac{d}{d\tau}\dot{x}_\mu = -\frac{e}{mc} F_{\mu\lambda}\dot{x}_\lambda$$

or

$$\frac{d}{d\tau}\dot{x} = -\frac{e}{mc}\boldsymbol{F}\dot{x}, \qquad \boldsymbol{F}=(F_{\mu\lambda}). \tag{A2}$$

The Hamilton-Jacobi equation is, on the other hand, given by

$$\frac{\partial S}{\partial \tau} + \frac{1}{2m}\left[\left(\frac{\partial S}{\partial x_\mu} + \frac{e}{2c}(x_\lambda - x_{0\lambda})F_{\lambda\mu}\right)^2 + m^2c^2\right] = 0. \tag{A3}$$

Instead of solving (A3) directly, we shall first handle the equation of motion (A2) which can easily be integrated because $\boldsymbol{F}$ is a constant matrix. Thus,

$$\dot{\boldsymbol{x}} = \exp\left[-\frac{e}{mc}\boldsymbol{F}\tau\right]\dot{\boldsymbol{x}}_0 \equiv \boldsymbol{S}\dot{\boldsymbol{x}}_0, \tag{A4}$$

and further

$$(\boldsymbol{x}-\boldsymbol{x}_0) = \left[(1-\boldsymbol{S}(\tau))\Big/\frac{e}{mc}\boldsymbol{F}\right]\dot{\boldsymbol{x}}_0,$$

$$\therefore \quad \dot{\boldsymbol{x}}_0 = \left[\frac{e\boldsymbol{F}}{mc}\Big/(1-\boldsymbol{S}(\tau))\right](\boldsymbol{x}-\boldsymbol{x}_0), \tag{A5}$$

$$\dot{\boldsymbol{x}} = \boldsymbol{S}\dot{\boldsymbol{x}}_0 = \left[\frac{e\boldsymbol{F}}{mc}\boldsymbol{S}\Big/(1-\boldsymbol{S})\right](\boldsymbol{x}-\boldsymbol{x}_0).$$

The conjugate momenta p_μ and the Hamiltonian $H=(p\dot{x})-L$ become

$$\begin{aligned}\boldsymbol{p} &= m\dot{\boldsymbol{x}} - (e/c)\boldsymbol{A} = m\dot{\boldsymbol{x}} + (e/2c)\boldsymbol{F}(\boldsymbol{x}-\boldsymbol{x}_0) \\ &= \left[\frac{e\boldsymbol{F}}{c}\boldsymbol{S}\Big/(1-\boldsymbol{S}) + \frac{e\boldsymbol{F}}{2c}\right](\boldsymbol{x}-\boldsymbol{x}_0) = \frac{e\boldsymbol{F}}{2c}(1+\boldsymbol{S})/(1-\boldsymbol{S})\cdot(\boldsymbol{x}-\boldsymbol{x}_0) \\ &= \frac{e}{2c}\boldsymbol{F}\operatorname{cth}\left(\frac{e}{2mc}\boldsymbol{F}\tau\right)\cdot(\boldsymbol{x}-\boldsymbol{x}_0),\end{aligned} \tag{A6}$$

$$\begin{aligned}H &= \frac{1}{2}mc^2 + \frac{1}{2m}\left(\boldsymbol{p}+\frac{e}{c}\boldsymbol{A}\right)^2 = \frac{1}{2}mc^2 + \frac{1}{2}m\dot{\boldsymbol{x}}^2 \\ &= \frac{1}{2}mc^2 + \frac{1}{2}mc^2\cdot\frac{e^2}{m^2c^2}\left(\frac{\boldsymbol{F}\boldsymbol{S}}{1-\boldsymbol{S}}\boldsymbol{x}-\boldsymbol{x}_0, \frac{\boldsymbol{F}\boldsymbol{S}}{1-\boldsymbol{S}}\boldsymbol{x}-\boldsymbol{x}_0\right) \\ &= \frac{1}{2}mc^2 + \frac{e^2}{2m}\left(\boldsymbol{x}-\boldsymbol{x}_0, \frac{-\boldsymbol{F}^2\boldsymbol{S}}{(1-\boldsymbol{S})^2}\boldsymbol{x}-\boldsymbol{x}_0\right),\end{aligned} \tag{A7}$$

using the relation for the transposed:

$$\tilde{\boldsymbol{S}} = \boldsymbol{S}^{-1}, \qquad \tilde{\boldsymbol{F}} = -\boldsymbol{F}. \tag{A8}$$

The action function S, which satisfies $\delta S = p\delta x - H\delta\tau$, is now obtained by putting

$$S = -\frac{1}{2}mc^2\tau + \frac{e}{4c}(\boldsymbol{x}-\boldsymbol{x}_0,\ \boldsymbol{G}(\boldsymbol{x}-\boldsymbol{x}_0)), \tag{A9}$$

$$\boldsymbol{G} = \boldsymbol{F}\,\mathrm{cth}\left(\frac{e}{2mc}\boldsymbol{F}\tau\right).$$

The remaining function f is determined by the equation

$$2m\frac{df}{d\tau} + \left(\Box^2 \boldsymbol{S} + \frac{ei}{2c}\sigma_{\mu\nu}F_{\mu\nu}\right)f = 0 \tag{A10}$$

From (A9),

$$\Box^2 \boldsymbol{S} = \frac{e}{2c}\,Tr\,\boldsymbol{G},$$

so that

$$f = \exp\left[-\frac{1}{2}\,Tr\left\{\ln \mathrm{sh}\left(\frac{e}{2mc}\boldsymbol{F}\tau\right) + \frac{ei}{2mc}(\sigma\boldsymbol{F})\tau\right\}\right]f_0. \tag{A11}$$

In these expressions, functions of the matrix $\boldsymbol{F}$ bear symbolical meaning. When it happens that det $|F|=0$ and consequently no inverse can be defined directly, we should go back to the beginning for the correct interpretation.

Now that the functions $F = \exp(iS/\hbar)f$ and A_μ are both even in the coordinates $x-x'$, we see that the argument proposed in Section 4 for the discussion of the induced current holds also in this general case.

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On the Convergence of the Perturbation Method, II. 1.

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Part II. Perturbation Method as Asymptotic Expansion

§ 1. Introduction

In a previous paper[1] we gave a proof of the convergence of the perturbation method (abr. p.m.) in the case of the so-called regular perturbation. As we already pointed out there, however, the assumption of the "regularity" is rather restrictive, and the theory could not comprise all important applications of the method, especially in problems of quantum mechanics.

In fact, it is too much to require the convergence in the mathematical sense of the formal series of p.m. In all practical problems only their first several terms are calculated, and they may ultimately be divergent, or even possessed of only finite terms, without being unavailable. Thus we are lead to regard them as asymptotic rather than power series, and it is natural to expect that the range of applicability of p.m. be much extended by this new interpretation.

It is the purpose of the present paper to show that this is really the case, and to derive several conditions under which the formal series are justified to represent the correct eigen-values and eigen-vectors within the specified order of approximation. Although is is often tedious to verify these conditions rigorously, they are of quite general character and may be regarded as sufficient to justify the use of p.m. in almost all practical problems.

As in I. our method is based on the operator theory of abstract Hilbert space. In contrast with the function-theoretical procedure of I., however, our argument is here closely related to the generalized variational principle developed in another paper.[2] We can even say that p.m. is essentially a special case of the variational method, where the approximate eigen-vector is so chosen that the possible error of the calculated eigen-value is "small" in the sense that it is formally of the possible highest order in the small parameter.

It will be convenient to give here a sketch of this relation. If H is the self-adjoint operator for which we are to solve the eigen-value problem, the cited variational principle consists in making the quantity

$$\varepsilon=\| (H-\eta) w\| / \| w\|$$

as small as possible by suitable choice of the number η and the vector w. For fixed w, the smallest ε is attained by taking $\eta=(Hw, w)/\|w\|^2$. Then it was shown[2] that the correct eigenvalue λ and eigen-vector φ are approximated by η and w within the possible errors of the order ε^2 and ε respectively. In the case of p.m. under consideration, H is represented as a formal power series of a small parameter x. If we take as w the *formal* series representing the eigen-vector of H, ε becomes *formally* zero. Consequently, if we choose as w the finite series obtained by taking the first n terms of this formal series, ε will be of the order x^n. According to what is just stated, it follows that η and w approximate the eigen-value λ and the eigen-vector φ within the possible errors of the order x^{2n} and x^n respectively. In other words, λ and φ will admit of asymptotic expansion *so long as the coefficients of the formal series are significant*, i.e. they can be calculated by operations within the Hilbert space.

The nature of our problem requires rigorous treatment, but this is not the place to enter into details of mathematical argument. So we are obliged to restrict ourselves in stating important conclusions, together with an outline of their derivation.

§ 2. Fundamental Assumptions and O-th Approximation

Let us consider an operator H_x of the (separable[3]) Hilbert space formally defined by

$$H_x = H_0 + xV, \tag{1}$$

where H and V are symmetric (or Hermitian) operators and x is a real, small parameter.* We assume that H_0 is self-adjoint[4] (i.e. the eigen-value problem is completely solvable for H_0). In other words, we have the spectral decomposition[4]

$$H_0 = \int_{-\infty}^{+\infty} \lambda \, dE_0(\lambda), \tag{2}$$

where $E_0(\lambda)$, a system of projection operators[5] called the resolution of the identity[4] belonging to H_0, is assumed to be known. We denote by D_0 the domain[6] of H_0. Further we assume

$$x \geqq 0 ; \tag{3}$$

this does not affect the generality, for if $x < 0$ we have only to consider $-V$ instead of V.

The problem of p.m. consists in developing the eigen-values and eigen-vectors

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* More general case

$$H_x = H_0 + xV' + x^2 V'' + \ldots\ldots$$

can be treated similarly, but for simplicity's sake we restrict ourselves to the case (1) in the following.

of $H_\varkappa$ as ascending series of $\varkappa$, the coefficients being expressed in terms of H_0 or $E_0(\lambda)$ and V. Clearly this is possible only when some conditions are imposed on thc relation between H_0 and V.

In the first place, the operator $H_0+\varkappa V$ is defined only in the common domain $\bar{D}$ of H_0 and V. Hence we must require that $\bar{D}$ is dense in the Hilbert space. Then $H_0+\varkappa V$ is clearly a symmetric operator. But this is not sufficient for the eigen-value problem to be solvable. For this purpose it must be possible to extend $H+\varkappa V$ to a self-adjoint operator *in an unambiguous way*. Practically this is possible only when one of the following conditions are satisfied:

i) $H_0+\varkappa V$ defined on $\bar{D}$ is essentially self-adjoint[7] for $\varkappa \geqq 0$*. Then $H_0+\varkappa V$ has a unique self-adjoint extension (namely its closure[8]), which will be denoted by $H_\varkappa$.

i′) Both H_0 and V are bounded below,[9] i.e. there are constants c_0, c such that

$$(H_0 f, f) \geqq c_0 (f, f),$$
$$(Vf, f) \geqq c\,(f, f).$$

Then the operator $H_0+\varkappa V$ defines a quadratic form $((H_0+\varkappa V)f,f)$ which is also bounded below. On applying the Ritz variational principle to this quadratic form, we can define in a unique way an extension of $H_0+\varkappa V$. *It is important that the operator thus determined is self-adjoint*, as was generally proved by Friedrichs.[10] We shall denote by $H_\varkappa$ this "Friedrichs extension" of $H_0+\varkappa V$. *In the following it is assumed that this $H_\varkappa$ coincides with H_0 for $\varkappa=0$.*

Of course it is rather usual that both i) and i′) are satisfied simultaneously. In each case the spectral decomposition of $H_\varkappa$ will be denoted by

$$H_\varkappa = \int_{-\infty}^{+\infty} \lambda\, dE_\varkappa(\lambda)\,. \tag{4}$$

Under these conditions we can deduce the important relation

$$E_\varkappa(\lambda) \to E_0(\lambda), \quad (\varkappa \to 0) \tag{5}$$

for all λ except for at most denumerable eigen-values** of H_0. Here the first → means strong convergence.*** The proof of (5), which is based on Stone's thorem[11] and, in the case of i′), also on Friedrichs's theorem[10], will be omitted here.

In the following we keep our eyes on a fixed, isolated eigenvalue λ_0 of H_0

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* For the immediate purpose we have only to require i) for $\varkappa > 0$. But it is necessary to include $\varkappa=0$ for later development.

** In this paper an eigen-value always means a "discrete" eigen-value, unless the contrary is positively stated.

*** That is, $\|E_\varkappa(\lambda)f - E_0(\lambda)f\| \to 0$ for every f. This is weaker than the "uniform convergence" used in I.

with finite multiplicity m. Then we can take two real numbers α, β such that $\alpha < \lambda_0 < \beta$ and that the closed interval $[\alpha, \beta]$ contains no other point of the spectrum of H_0. It means that the projection operator

$$E_0 = E_0(\beta) - E_0(\alpha) \tag{6}$$

coincides with the projection on the eigen-space of H_0 associated with λ_0. We have by (5) and (6)

$$E_\kappa(\beta) - E_\kappa(\alpha) \rightarrow E_0, \quad (\kappa \rightarrow 0). \tag{7}$$

It follows easily that

$$\dim\,(E_\kappa(\beta) - E_\kappa(\alpha)) \geqq \dim E_0 = m \tag{8}$$

for sufficiently small κ. This inequality means that the interval $(\alpha, \beta]$ contains either at least m eigen-values (considering the degeneracy) of H_κ or points of the continuous spectrum of H_κ.

For our purpose, however, it is necessary to have equality sign in (8). For otherwise the multiplicity of λ_0 would abruptly increase or λ_0 would diffuse into continuous spectrum when the perturbation κV is switched on, and p.m. would be invalidated. Thus we are obliged to introduce the following further assumption*:

ii) α, β can be chosen such that

$$\dim\,(E_\kappa(\beta) - E_\kappa(\alpha)) \leqq m \tag{9}$$

for sufficiently small κ.

Then we have, on combining (8) and (9)

$$\dim\,(E_\kappa(\beta) - E_\kappa(\alpha)) = m, \tag{10}$$

which assures that the interval $(\alpha, \beta]$ contains just m eigenvalues of H_κ and no other points of its spectrum. We denote these eigen-values by

$$\lambda_{\kappa 1} \leqq \lambda_{\kappa 2} \leqq \cdots\cdots \leqq \lambda_{\kappa m}. \tag{11}$$

Now let α', β' be such that $\alpha < \alpha' < \lambda_0 < \beta' < \beta$. Then (9) holds for α', β' *a fortiori*. Also (8) is true for α', β' provided κ is sufficiently small. Thus we obtain (10) also for α', β', showing that the eigen-values (11) are contained in $(\alpha', \beta']$ for sufficiently small κ. Since the interval (α', β') can be chosen arbitrarily small, it follows that

$$\lambda_{\kappa i} \rightarrow \lambda_0, \ (\kappa \rightarrow 0, \ i = 1, \ \ldots\ldots, \ m). \tag{12}$$

Thus the eigen-values under consideration are continuous at $\kappa = 0$. The same is true with (the projection operator on) the total m-dimensional subspace determined by the corresponding m-eigenvectors**, as we see directly from (7). In this sense we can say that *p.m. is valid to the* 0*-th order approximation under the assumptions i), ii) or i′), ii).*

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* However, we shall later be able to relax this restriction to some extent. See § 6.

** This is not necessarily the case, under the present assumptions alone, with the eigen-vector for *each* $\lambda_{\kappa i}$ $(i = 1, \ \ldots\ldots, \ m)$ *separately*, if $m > 1$.

These conditions are in any case necessary* for p.m. to be valid, and in this sense they are not restrictive. Also they can be shown to be satisfied in almost all problems of application. But it must be confessed that they are often very difficult to verify rigorously. It would be desirable to replace them by simpler conditions, but as we failed in doing so without sacrificing generality, we shall assume them in the following and, as a rule, leave their verification to individual cases.

For completeness' sake, however, we shall mention two special, but important, class of cases where these conditions are generally shown to be satisfied:

A) i), ii) are satisfied if there are constants a, b such that

$$\|Vf\| \leq a\|H_0 f\| + b\|f\|.$$

This is the case treated in I., § 4, and the perturbation is even regular, formal series being really *convergent.*

B) ii) is automatically satisfied if i′) holds and, moreover, the spectrum of H_0 *below* λ_0 consists of pure point spectrum.

On the other hand, there are a few examples where these conditions are not satisfied. One of the most remarkable cases is the Hamiltonian operotor of the Stark effect of the hydrogenic atom, where i) is satisfied but ii) is not because the perturbed operator has no eigen-value at all. In order to justify the application of p.m. to such a problem, more profound study is necessary than that given in this paper. This is namely the case of "weak quantization," and can be treated properly only in connection with the theory of p.m. containing the time. We hope to discuss this problem in due course.

§ 3. First and Second Approximations

In the preceding section we have shown that p.m. is valid at least to the O-th approximation under very general conditions. But this is, though necessary, not sufficient for practical purposes. In this and the next sections we shall give sufficient conditions for higher approximations to be valid. The fundamental conditions i), ii) or i′), ii) are always assumed in the following.

For simplicity's sake, we shall for the present assume that the eigen-value λ_0 is non-degenerate, i.e. $m=1$. Accordingly we write simply $\lambda_\varkappa$ in place of $\lambda_{\varkappa 1}$. Let φ_0 and $\varphi_\varkappa$ be the associated normalized eigen-vectors of H_0 and $H_\varkappa$ respectively.

Then the formal series of p.m. are given by[12]

$$\lambda_\varkappa=\lambda_0+\varkappa\lambda^{(1)}+\varkappa^2\lambda^{(2)}+\varkappa^3\lambda^{(3)}+\cdots, \tag{13}$$

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* See, however, the foregoing footnote.

$$\varphi_\varkappa=\varphi_0+\varkappa\varphi^{(1)}+\varkappa^2\varphi^{(2)}+\cdots, \tag{14}$$

where

$$\left.\begin{aligned}&\lambda^{(1)}=(V\varphi_0,\ \varphi_0),\\&\lambda^{(2)}=-(SV\varphi_0,\ V\varphi_0),\\&\lambda^{(3)}=(VSV\varphi_0,\ SV\varphi_0)-\lambda^{(1)}\|SV\varphi_0\|^2,\\&\cdots\cdots\cdots\cdots\cdots\end{aligned}\right\} \tag{15}$$

$$\left.\begin{aligned}&\varphi^{(1)}=-SV\varphi_0,\\&\varphi^{(2)}=SVSV\varphi_0-\frac{1}{2}\|SV\varphi_0\|^2\varphi_0-\lambda^{(1)}S^2V\varphi_0,\\&\cdots\cdots\cdots\cdots\cdots\end{aligned}\right\} \tag{16}$$

with

$$S=\int'(\lambda-\lambda_0)^{-1}\,dE_0(\lambda), \tag{17}$$

$\int'$ indicating that the isolated eigen-value λ_0 be excluded from the integration. We note that S is a bounded[13)] operator and hence can be operated on every vector of the Hilbert space, and further that

$$S\varphi_0=0. \tag{18}$$

It should be noted that these expressions are quite formal and meaningless unless we introduce some further assumptions. We shall do this step by step.

First we assume that $V\varphi_0$ exists, i.e. φ_0 belongs to the domain of V. Then, since S is bounded, the three coefficients $\lambda^{(1)}$, $\lambda^{(2)}$ and $\varphi^{(1)}$ are certainly significant, and we assert

$$\left.\begin{aligned}&\lambda_\varkappa=\lambda_0+\varkappa\lambda^{(1)}+\varkappa^2\lambda^{(2)}+o(\varkappa^2),\\&\varphi_\varkappa=\varphi_0+\varkappa\varphi^{(1)}+o(\varkappa).\end{aligned}\right\} \tag{19}$$

Thus p.m. is valid in this case to the 2nd approximation with respect to the eigen-value and to the 1st approximation with respect to the eigen-vector.

Complete proof of (19) is not very simple, and here we must content ourselves with the proof of somewhat less precise result.

We consider φ_0 as an approximate eigen-vector of $H_\varkappa$ and apply the theorem of E., § 2. For this purpose we have to calculate

$$\eta_\varkappa=(H_\varkappa\varphi_0,\ \varphi_0)=\lambda_0+\varkappa\lambda^{(1)},$$

$$\varepsilon_\varkappa=\|(H_\varkappa-\eta_\varkappa)\varphi_0\|=\varkappa\|(V-\lambda^{(1)})\varphi_0\|.$$

According to the cited theorem, we have therefore

$$\left.\begin{aligned}&|\lambda_\varkappa-\lambda_0-\varkappa\lambda^{(1)}|\leq\varepsilon_\varkappa^2d^{-1}=\varkappa^2\|(V-\lambda^{(1)})\varphi_0\|^2d^{-1},\\&\|\varphi_\varkappa-\varphi_0\|\leq\varepsilon_\varkappa d^{-1}=\varkappa\|(V-\lambda^{(1)})\varphi_0\|d^{-1},\end{aligned}\right\} \tag{20}$$

where d is a lower bound to the quantities $\eta_\varkappa-\alpha$ and $\beta-\eta_\varkappa$, and can be chosen as independent of $\varkappa$ on account of (10). (20) gives $\lambda_{\varkappa}$ correctly to the 1st and $\varphi_{\varkappa}$ to the O-th order in $\varkappa$,* and is somewhat less precise than the asserted relation (19). In order to prove the latter, we must take better approximate eigenvector than φ_0 used sbove, but it is rather complicated since we cannot take the next approximation $\varphi_0+\varkappa\varphi^{(1)}$ directly, for we did not as yet assume the existence of $V\varphi^{(1)}$**. We give up to enter into these details here.

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[*] It is the advantage of our method that it gives us not only the order of magnitude, but also a rigorous estimates, of the errors involved, as is seen from (20).

[**] If $V\varphi^{(1)}$ or $-VSV\varphi_0$ is assumed to exist, $\lambda_{\varkappa}$ will be given correctly up to $\varkappa^4$, as we shall show in the next section.

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On the Nuclear Forces.*

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Summary.

Generalizing the Lagrangian of the interaction as has been done by Schoenberg, and assuming the relations between coupling constants in the symmetrical and charged spin 0 meson fields, nuclear forces which have the same character as the Bethe s potential by the neutral vector meson theory does will be given. There is, however, no difference between the usual scalar-pseudoscalar mixed theory and our generalized interaction one in the approximation adopted here.

§ 1. Introduction.

In studying the interaction of mesons with nucleons the pseudoscalar meson theory has been considered by several authors. For many years the central problem of the meson theory has been to look for a satisfactory type of the meson field to explain the properties of nuclei by the customary perturbation method. As is well-known, definite advances towards the understanding of the divergence problem have been made. Recently, using the newly developed method Case[1] and others[2] showed that the additional nuclear magnetic moments calculated by the pseudoscalar meson theory in the Tomonaga-Schwinger formalism could not fit the existing experimental data in all cases, namely, neutral, charged and symmetrical theories. This point is the serious difficulties for the present meson theory. It must be, however, noted that, although it is hardly possible for us to decide the type of the meson which interacts with nucleons by investigating only the nuclear magnetic moment, still, the difficulty is the important disproof for the present meson theory.

On the other hand, considering that many results obtained by the pseudoscalar meson field have been in agreement with various experimental data, one could not discard it at once. In the present situation there seems to be another problem on the pseudoscalar meson theory. That is the Dyson's theorem[2] concerning the interactions between mesons and nucleons. Dyson has shown that the pseudoscalar interaction and the pseudovector one are completely equivalent for the first order radiative process. It is suggested from his argument that we should call our attentions not only to the procedure of the recent theory of the quantum electrodynamics, but also to the interaction schemes between mesons and nucleons. That is to say, as the interaction between meson fields and nuclear particles differs in many respects from that between electromagnetic field and electrons, it seems necessary that the meson theory should be throughly investigated from the aspect of interactions.

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* Preliminary report of the results appeared in Prog. Theor. Phys. 4 (1949), 572.

It may be regarded, further, as a feature of his point of view that, contrary to the previous treatments, where the nuclear forces came out from the non-relativistic Hamiltonians of interactions, the whole effects of relativistic terms should be taken into account at the outset of calculations.

Under these circumstances the problems of the meson and nuclear forces must be considered from the new stand point of view.

We shall treat here the nuclear forces exclusively, and nuclear magnetic moments and the electric quadrupole moment of a deuteron are being under investigation.

Now, in order to modify the present pseudoscalar meson theory, the following points are taken into account:

(1) Besides π mesons there exist τ mesons which appear frequently in the cosmic ray phenomena. But, the estimated coupling constant[3) between the τ meson and the nucleon is smaller (about 1/10) than that between the π meson and the nucleon. Further the force range of the τ meson is rather short owing to its large mass. Hence we are able to neglect the effect of the τ meson for nuclear potentials.

(2) Case[1) pointed out that the pseudoscalar meson theory is peculiar in that the meson cloud is particularly closely bound to the nucleon, and suggested the idea of trying to seek some model which gives rough qualitative agreement with the experiment. According to his calculation the scalar meson theory leads to the definite result for the magnetic moment, therefore the so-called mixed theory will have the finite one.

Based on the foothold of the pseudoscalar meson theory, we should like to consider the nuclear potential by mixing the scalar field with it or generalizing the Lagrangians of interactions as will be shown in the next section. The purpose of this paper is to examine the nuclear forces which will be obtained in the latter direction.

In §2 we briefly describe the essential point of Schoenberg's[4) idea, namely, the extension of the Kemmer's[5) Lagrangians for the interaction.

Then in §3, making use of the canonical transformation which has been done by van Hove,[6) we obtain the second order nuclear potentials.

Taking into account the Dirac equation for the two body problem,[7) we get usual potentials in §4. Finally in §5 we compare our results with the Bethe's potential by the neutral vector meson theory.

§ 2. General theories of spin 0 and spin 1 particles.

According to Kemmer[5] the most general interaction must satisfy the following conditions:

(1) The interaction term to be added to the Lagrangian involves only the wave function of the nucleons without their derivatives. (2) The field quantities which enter the interaction expression are the ϕ and χ only, derivatives of these quantities not occurring explicitly.

For these assumptions Kemmer showed four types of mesons, Schoenberg[4] proposed, however, generalizations of the Kemmer's interactions by introducing the pseudoscalar $\eta^{\alpha\beta\gamma\delta}$

$$\eta^{\alpha\beta\gamma\delta}=(-g)^{1/2}\,\epsilon^{\alpha\beta\gamma\delta},\quad g=|g_{\alpha\beta}|,\quad (\alpha,\beta,\gamma,\delta=,0,1,2,3), \tag{1}$$

$\epsilon^{\alpha\beta\gamma\delta}$ is the well-known Eddington's[9] tensor density, antisymmetric in the four indices and equal to ±1, according to the parity of permutation $\alpha,\beta,\gamma,\delta$ of the numbers 0, 1, 2, 3. Thus, for his interactions there are only two types of mesons, i.e. spin 0 meson and spin 1 meson. The explicit representation of his idea is as follows (field equations and Lagrangians of interactions for the neutral meson field). ($\hbar=c=1,\ x_0=t,\ x_1=x,\ x_2=y,\ x_3=z$).

(a) *Scalar field.*

$$\left\{\begin{aligned} &\frac{\partial\phi}{\partial x_\alpha}=\varkappa_0\chi_\alpha && (2)\\ &\frac{\partial\chi^\alpha}{\partial x_\alpha}=\varkappa_0\phi && (3)\end{aligned}\right.$$

$$L_a=-f_1 w\phi-f_2 v_\alpha\chi^\alpha, \tag{4}$$

$$L'_a=-\frac{1}{24}f'_1 s_{\alpha\beta\gamma\delta}\,\eta^{\alpha\beta\gamma\delta}\phi-\frac{1}{6}f'_2 t_{\alpha\beta\gamma}\,\eta^{\alpha\beta\gamma\delta}\chi_\delta, \tag{5}$$

$$(\phi=U,\quad \chi_0=-V_0,\quad \chi_i=V),$$

$$(i,k=1,2,3).$$

(b) *Pseudoscalar field.*

$$\left\{\begin{aligned} &\frac{\partial\phi_{\beta\gamma\delta}}{\partial x_\alpha}-\frac{\partial\phi_{\alpha\gamma\delta}}{\partial x_\beta}+\frac{\partial\phi_{\alpha\beta\delta}}{\partial x_\gamma}-\frac{\partial\phi_{\alpha\beta\gamma}}{\partial x_\delta}=\varkappa_0\chi_{\alpha\beta\gamma\delta} && (6)\\ &\frac{\partial\chi^{\alpha\beta\gamma\delta}}{\partial x_\alpha}=\varkappa_0\phi^{\beta\gamma\delta}, && (7)\end{aligned}\right.$$

$$(\phi_{\alpha\beta\gamma}=-\phi_{\beta\alpha\gamma}=-\phi_{\alpha\gamma\beta}),$$

$$L'_d=-\frac{1}{24}f'_1 s_{\alpha\beta\gamma\delta}\,\chi^{\alpha\beta\gamma\delta}+\frac{1}{6}f'_2 t_{\alpha\beta\gamma}\,\phi^{\alpha\beta\gamma}, \tag{8}$$

$$L_d=\frac{1}{24}f_1 w\,\eta^{\alpha\beta\gamma\delta}\chi_{\alpha\beta\gamma\delta}+\frac{1}{6}f_2 v_\alpha\,\eta^{\alpha\beta\gamma\delta}\phi_{\beta\gamma\delta}, \tag{9}$$

$$(\chi^{0123}=U,\quad \phi^{123}=-V_0,\quad \phi^{0ik}=-\mathbf{V}).$$

(c) *Vector field.*

$$\begin{cases} \dfrac{\partial\phi_\beta}{\partial x_\alpha}-\dfrac{\partial\phi_\alpha}{\partial x_\beta}=\varkappa_1\chi_{\alpha\beta} & (10) \\[2ex] \dfrac{\partial\chi^{\alpha\beta}}{\partial x_\alpha}=\varkappa_1\phi^\beta, & (11) \end{cases}$$

$$L_b=g_1 v_\alpha\phi^\alpha+\frac{1}{2}g_2 u_{\alpha\beta}\chi^{\alpha\beta}, \tag{12}$$

$$L_b'=+\frac{1}{6}g_1' t_{\alpha\beta\gamma}\eta^{\alpha\beta\gamma\delta}\phi_\delta+\frac{1}{4}g_2' u_{\alpha\beta}\eta^{\alpha\beta\gamma\delta}\chi_{\gamma\delta}, \tag{13}$$

$$(\phi_0=U_0,\quad \phi_i=-\mathbf{U},\quad \chi_{0i}=\mathbf{F},\quad \chi_{ik}=-\mathbf{G})$$

(d) *Pseudovector field.*

$$\begin{cases} \dfrac{\partial\phi_{\beta\gamma}}{\partial x_\alpha}-\dfrac{\partial\phi_{\alpha\gamma}}{\partial x_\beta}+\dfrac{\partial\phi_{\alpha\beta}}{\partial x_\gamma}=\varkappa_1\chi_{\alpha\beta\gamma} & (14) \\[2ex] \dfrac{\partial\chi^{\alpha\beta\gamma}}{\partial x_\alpha}=\varkappa_1\phi^{\beta\gamma}, & (15) \end{cases}$$

$$(\phi_{\alpha\beta}=-\phi_{\beta\alpha}),$$

$$L_c'=-\frac{1}{6}g_1' t_{\alpha\beta\gamma}\chi^{\alpha\beta\gamma}-\frac{1}{2}g_2' u_{\alpha\beta}\phi^{\alpha\beta}, \tag{16}$$

$$L_c=-\frac{1}{6}g_1 v_\alpha\eta^{\alpha\beta\gamma\delta}\chi_{\beta\gamma\delta}+\frac{1}{4}g_2 u_{\alpha\beta}\eta^{\alpha\beta\gamma\delta}\phi_{\gamma\delta}, \tag{17}$$

$$(\chi_{123}=U_0,\quad \chi_{0ik}=-\mathbf{U},\quad \phi_{ik}=-\mathbf{F},\quad \phi_{0i}=-\mathbf{G}),$$

in which the source densities and the Dirac matrices are defined by the following equations:

$$\left.\begin{aligned} &w=\psi^*\rho_3\psi,\quad (v_0,\ v_i)=(-\psi^*\psi,\ \psi^*\alpha\psi),\\ &(u_{0i},\ u_{ik})=(\psi^*\rho_2\boldsymbol{\sigma}\psi,\ \psi^*\rho_3\boldsymbol{\sigma}\psi),\\ &(t_{0ik},\ t_{ikl})=(\psi^*\boldsymbol{\sigma}\psi,\ -\psi^*\gamma^5\psi),\\ &s_{0123}=\psi^*\rho_2\psi, \end{aligned}\right\} \tag{18}$$

$$\gamma^i=-i\rho_3\alpha_i,\quad \sigma_l=-i\alpha_i\alpha_k\quad (i,\ k,\ l:\ cyclic),\quad \gamma^5=-\gamma^1\gamma^2\gamma^3\gamma^4.$$

As regards notations, we shall here give a brief survey of them which will be used extensively in the following.

Among these expressions, (4), (8), (12) and (16) are the usual Kemmer's interactions, and others are terms to be added to them for the generalization. f_i' and g_i' $(i=a,\ b,\ c,\ d.)$ are the constants which are concerned with the odd character of source densities in the usual meaning. These coupling constants have each the dimension of an electric charge.

We have shown also the vector notation for quantities of meson fields for each case.

Then, at the first place, let us revert to vector notation for the field equations. From above expressions we can obtain

$$\left.\begin{aligned} &\varkappa_0 \boldsymbol{V} = \text{grad}\, U, && \varkappa_0 V_0 = -\frac{\partial U}{\partial t}, \\ &\text{div}\, \boldsymbol{V} + \frac{\partial V_0}{\partial t} = \varkappa_0 U, && \text{curl}\, \boldsymbol{V} = 0, \end{aligned}\right\} \tag{19}$$

for (2) and (3), and (6) and (7). For the vector field we can put similarly

$$\left.\begin{aligned} &\varkappa_1 \boldsymbol{F} = -\frac{\partial \boldsymbol{U}}{\partial t} - \text{grad}\, U_0, && \varkappa_1 \boldsymbol{G} = \text{curl}\, \boldsymbol{U}, \\ &\frac{\partial \boldsymbol{F}}{\partial t} - \text{curl}\, \boldsymbol{G} = \varkappa_1 \boldsymbol{U}, && \text{div}\, \boldsymbol{F} + \varkappa_1 U_0 = 0. \end{aligned}\right\} \tag{20}$$

In the second place, we shall be concerned with the Lagrangians of interactions. On the contrary to the Schoenberg's notation, coupling constants are chosen so that they might lead to the same Lagrangians in the vecror notation for the spin 0 and spin 1 fields respectively. That is, the Lagrangians are

$$\begin{aligned} L_0 &= L_a + L_a' = L_d + L_d' \\ &= -\psi^* \left[f_1 \rho_3 U + \frac{f_2}{\varkappa_0} \left(\frac{\partial U}{\partial t} + \boldsymbol{\alpha} \cdot \text{grad}\, U \right) \right] \psi \\ &\quad - \psi^* \left[f_1' \rho_2 U + \frac{f_2'}{\varkappa_0} \left(\rho_1 \frac{\partial U}{\partial t} + \boldsymbol{\sigma} \cdot \text{grad}\, U \right) \right] \psi, \end{aligned} \tag{21}$$

for the spin 0 field, and

$$\begin{aligned} L_1 &= L_b + L_b' = L_c + L_c' \\ &= -\psi^* \left[g_1 (\boldsymbol{\alpha} \cdot \boldsymbol{U} - U_0) + \frac{g_2}{\varkappa_1} \left\{ -\rho_2 \boldsymbol{\sigma} \cdot \left(\frac{\partial \boldsymbol{U}}{\partial t} + \text{grad}\, U_0 \right) + \rho_3 \boldsymbol{\sigma} \cdot \text{curl}\, \boldsymbol{U} \right\} \right] \psi \\ &\quad - \psi^* \left[g_1' (\boldsymbol{\sigma} \cdot \boldsymbol{U} - \rho_1 U_0) + \frac{g_2'}{\varkappa_1} \left\{ \rho_3 \boldsymbol{\sigma} \cdot \left(\frac{\partial \boldsymbol{U}}{\partial t} + \text{grad}\, U_0 \right) + \rho_2 \boldsymbol{\sigma} \cdot \text{curl}\, \boldsymbol{U} \right\} \right] \psi, \end{aligned} \tag{22}$$

for the spin 1 field.

The Lagrangians for the free nucleon and the free meson are assumed to be of the forms

$$L_N = -\left(\frac{1}{2i}\right) \Big[\psi^* (\boldsymbol{\alpha} \cdot \text{grad}\, \psi) - (\text{grad}\, \psi^* \cdot \boldsymbol{\alpha}) \psi + \psi^* \frac{\partial \psi}{\partial t} - \frac{\partial \psi^*}{\partial t} \psi + 2iM \psi^* \rho_3 \psi \Big], \Big)$$

$$\left.\begin{aligned} L_{M_0}&=-\left(\frac{1}{2}\right)\left[(\mathrm{grad}\,U\cdot\mathrm{grad}\,U)+\varkappa_0^2U^2-\left(\frac{\partial U}{\partial t}\right)^2\right], \\ L_{M_1}&=-\left(\frac{1}{2}\right)\left[(\mathrm{curl}\,\boldsymbol{U}\cdot\mathrm{curl}\,\boldsymbol{U})-\left(\frac{\partial \boldsymbol{U}}{\partial t}+\mathrm{grad}\,U_0\right)\cdot\left(\frac{\partial \boldsymbol{U}}{\partial t}+\mathrm{grad}\,U_0\right)\right. \\ &\qquad\qquad\left.+\varkappa_1^2(\boldsymbol{U}\cdot\boldsymbol{U}-U_0^2)\right]. \end{aligned}\right\} \tag{23}$$

Thus, the total ones are given by

$$L(i)=L_N+L_{M_i}+L_i, \qquad (i=0,\ 1). \tag{24}$$

Defining the canonical conjugate Π_0 and Π_1 in the usual way, we obtain

$$\Pi_0=\frac{\delta L(0)}{\delta\left(\dfrac{\partial U}{\partial t}\right)}=\frac{\partial U}{\partial t}-\frac{f_2}{\varkappa_0}\,\psi^*\psi-\frac{f_2'}{\varkappa_0}\,\psi^*\rho_1\psi \tag{25}$$

and

$$\Pi_1=\frac{\delta L(1)}{\delta\left(\dfrac{\partial \boldsymbol{U}}{\partial t}\right)}=\left(\frac{\partial \boldsymbol{U}}{\partial t}+\mathrm{grad}\,U_0\right)+\frac{g_2}{\varkappa_1}\,\psi^*\rho_2\sigma\psi-\frac{g_2'}{\varkappa_1}\,\psi^*\rho_3\sigma\psi. \tag{26}$$

We can now pass to the Hamiltonians by the usual procedure.

(*Spin* 0 *field*)

$$H(0)=H_N+H_{M_0}+H_0'+H_{0\ \mathrm{dir.}}, \tag{27}$$

$$H_N=\left(\frac{1}{2i}\right)\int[\psi^*(\boldsymbol{\alpha}\cdot\mathrm{grad}\,\psi)-(\mathrm{grad}\,\psi^*\cdot\boldsymbol{\alpha})\psi+2iM\psi^*\rho_3\psi]\,dV, \tag{27·1}$$

$$H_{M_0}=\left(\frac{1}{2}\right)\int[(\mathrm{grad}\,U\cdot\mathrm{grad}\,U)+\varkappa_0^2\,U^2+\Pi_0^2]\,dV, \tag{27·2}$$

$$\begin{aligned} H_0'=\int\psi^*\Big[f_1\rho_3U+f_1'\rho_2U+\frac{f_2}{\varkappa_0}\,(\boldsymbol{\alpha}\cdot\mathrm{grad}\,U)+\frac{f_2'}{\varkappa_0}(\boldsymbol{\sigma}\cdot\mathrm{grad}\,U) \\ +\frac{f_2}{\varkappa_0}\,\Pi_0+\frac{f_2'}{\varkappa_0}\,\Pi_0\rho_1\Big]\psi\,dV. \end{aligned} \tag{27·3}$$

$$\begin{aligned} H_{0,\mathrm{div.}}=\frac{1}{2}\iint\Big[\left(\frac{f_2}{\varkappa_0}\right)^2(\psi^*\psi)^2+\left(\frac{f_2'}{\varkappa_0}\right)^2(\psi^*\rho_1\psi)^2+\left(\frac{f_2f_2'}{\varkappa_0^2}\right)(\psi^*\psi)(\psi^*\rho_1\psi) \\ +\left(\frac{f_2'f_2}{\varkappa_0^2}\right)(\psi^*\rho_1\psi)(\psi^*\psi)\Big]\,dV. \end{aligned} \tag{27·4}$$

(*Spin* 1 *field*)

$$H(1)=H_N+H_{M_1}+H_1'+H_{1,\mathrm{dir.}}, \tag{28}$$

$$H_N=(27\cdot1), \tag{28·1}$$

$$H_{M_1}=\left(\frac{1}{2}\right)\int\Big[(\mathrm{curl}\,\boldsymbol{U}\cdot\mathrm{curl}\,\boldsymbol{U})+\kappa_1^2(\boldsymbol{U}\cdot\boldsymbol{U})+\left(\frac{1}{\kappa_1^2}\right)(\mathrm{div}\,\boldsymbol{\Pi}_1)^2$$
$$+(\boldsymbol{\Pi}_1\cdot\boldsymbol{\Pi}_1)\Big]dV, \tag{28·2}$$

$$H_1'=\int\psi^*\Big[g_1\Big(\boldsymbol{\alpha}\cdot\boldsymbol{U}-\frac{1}{\kappa_1^2}\,\mathrm{div}\,\boldsymbol{\Pi}_1\Big)+\frac{g_2}{\kappa_1}(\rho_3\boldsymbol{\sigma}\cdot\mathrm{curl}\,\boldsymbol{U}-\rho_2\boldsymbol{\sigma}\,\boldsymbol{\Pi}_1)$$
$$+g_1'\Big(\boldsymbol{\sigma}\cdot\boldsymbol{U}-\frac{1}{\kappa_1^2}\,\rho_1\,\mathrm{div}\,\boldsymbol{\Pi}_1\Big)+\frac{g_2'}{\kappa_1}(\rho_2\boldsymbol{\sigma}\cdot\mathrm{curl}\,\boldsymbol{U}+\rho_3\boldsymbol{\sigma}\,\boldsymbol{\Pi}_1)\Big]\psi\,dV, \tag{28·3}$$

$$H_{1,\,\mathrm{dir.}}=\frac{1}{2}\int\Big[\Big(\frac{g_1}{\kappa_1}\Big)^2(\psi^*\psi)^2+\Big(\frac{g_2}{\kappa_1}\Big)^2(\psi^*\rho_2\boldsymbol{\sigma}\psi)^2+\Big(\frac{g_1'}{\kappa_1}\Big)^2(\psi^*\rho_1\psi)^2$$
$$+\Big(\frac{g_2'}{\kappa_1}\Big)^2(\psi^*\rho_3\boldsymbol{\sigma}\psi)^2+\Big(\frac{g_1g_1'}{\kappa_1^2}\Big)\{(\psi^*\rho_1\psi)(\psi^*\psi)+(\psi^*\psi)(\psi^*\rho_1\psi)\}$$
$$-\Big(\frac{g_2g_2'}{\kappa_1^2}\Big)\{(\psi^*\rho_3\boldsymbol{\sigma}\psi)(\psi^*\rho_2\boldsymbol{\sigma}\psi)+(\psi^*\rho_2\boldsymbol{\sigma}\psi)(\psi^*\rho_3\boldsymbol{\sigma}\psi)\}\Big]dV, \tag{28·4}$$

where $H_{i,\,\mathrm{dir.}}$ $(i=0,1)$ are contact interaction terms.

§ 3. Second order interactions.

By virtue of canonical transformations which have been used by van Hove,[6] the terms H_i' of the first order in coupling constants are carried over to the second order. Here we shall adopt the same notations as van Hove's, and consider the spin 0 field and spin 1 field separately. Let us write down the matrix elements of H_i'' for a transition from a initial state Φ to a final state Ψ of the system.

$$\begin{cases}\Phi=\dfrac{1}{\sqrt{2}}\sum\limits_{m_1,m_2}\Phi_{m_1m_2}\,\psi^{(1)}_{m_1}\,\psi^{(2)}_{m_2}\,,\\[2ex] \Psi=\dfrac{1}{\sqrt{2}}\sum\limits_{n_1,n_2}\Psi_{n_1n_2}\,\psi^{(1)}_{n_1}\,\psi^{(2)}_{n_2}\,.\end{cases} \tag{29}$$

(*Spin 0 field*)

The Fourier expansion of the meson field is

$$U(x,x_0)=\sum_k\frac{1}{\sqrt{2k_0}}[U(k)\exp\{i(\boldsymbol{k}\cdot\boldsymbol{x}-k_0x_0)\}$$
$$+U^*(k)\exp\{i(-\boldsymbol{k}\cdot\boldsymbol{x}+k_0x_0)\}], \tag{30}$$

where

$$k_0^2=k^2+\kappa_0^2,\qquad k_0=\sqrt{k^2+\kappa_0^2}\,. \tag{31}$$

After some computations the transformed Hamiltonian leads to

$$e^{iS^0}H(0)\,e^{-iS^0}=H_N+H_{M_0}+H_{0,\mathrm{dir.}}+H_0'', \tag{32}$$

$$H_0''=\sum\Psi_{n_1n_2}\,\chi^{(1)*}_{n_1}\,\chi^{(2)*}_{n_2}\,(H_{p.s.}+H_{s.}+H^0_{m.})\,\chi^{(1)}_{m_1}\,\chi^{(2)}_{m_2}\,\Phi_{m_1m_2}, \tag{33}$$

$$H_{p.s.}=-\left(\frac{f_1'}{\varkappa_0}\right)^2\rho_1^{(1)}\rho_1^{(2)}+f_3^2\frac{\rho_2^{(1)}\rho_2^{(2)}}{(E_{m_1}-E_{n_1})^2-k_0^2}-i\left(\frac{f_2'f_3'}{\varkappa_0}\right)\frac{(\rho_1^{(1)}\rho_2^{(2)}+\rho_2^{(1)}\rho_1^{(2)})(E_{m_1}-E_{n_1})}{(E_{m_1}-E_{n_1})^2-k_0^2}, \tag{34}$$

$$H_{s.}=-\left(\frac{f_2}{\varkappa_0}\right)^2+f_1^2\frac{\rho_3^{(1)}\rho_3^{(2)}}{(E_{m_1}-E_{n_1})^2-k_0^2}-i\left(\frac{f_1f_2}{\varkappa_0}\right)\frac{(\rho_3^{(1)}+\rho_3^{(2)})(E_{m_1}-E_{n_1})}{(E_{m_1}-E_{n_1})^2-k_0^2}, \tag{35}$$

$$H_{m.}^0=-\left(\frac{f_2f_2'}{\varkappa_0^2}\right)(\rho_1^{(1)}+\rho_1^{(2)})+f_1f_3'\frac{(\rho_2^{(1)}\rho_3^{(2)}+\rho_3^{(1)}\rho_2^{(2)})}{(E_{m_1}-E_{n_1})^2-k_0^2}$$

$$-i\left(\frac{f_2'f_1}{\varkappa_0}\right)\frac{(\rho_3^{(1)}\rho_1^{(2)}+\rho_3^{(2)}\rho_1^{(1)})(E_{m_1}-E_{n_1})}{(E_{m_1}-E_{n_1})^2-k_0^2}-i\left(\frac{f_2f_3'}{\varkappa_0}\right)\frac{(\rho_2^{(1)}+\rho_2^{(2)})(E_{m_1}-E_{n_1})}{(E_{m_1}-E_{n1})^2-k_0^2}, \tag{36}$$

where

$$f_3'=f_1'+2\left(\frac{M}{\varkappa_0}\right)f_2', \tag{37}$$

E_m, E_n: Initial and final energies of nucleons.

and $H_{p.s.}$ and $H_{s.}$ are usual pseudoscalar and scalar terms, but H_m^0 is the mixed term of them which comes from the generalized interaction. It is the remarkable fact in this expression that (34) and (35) just coincide with the Nambu's[10] result by making use of the relativistic formulation of perturbation theory* except the contact interaction terms. Then, the so-called vector interaction of scalar meson vanishes completly in (34). Hence Case's[1] result holds also in our generalized formalism.

As can be seen immediately, there exist perfect correspondences between the contact interaction terms of the initial Hamiltonian and that of transformed one, and they cancel out each other. In other words there is no ambiguity of the contact interaction terms which has been in the derivation of nuclear potentials by the perturbation method. Of course, it is the well-known fact in the Tomonaga-Schwinger formalism that when nuclear forces are restricted to the seeond order in coupling constants, there arise cancellations for scalar and pseudoscalar fields respectively as shown by Case.

(*Spin 1 field*)

We first write the Fourier expansions of the meson field as before:

$$U_0(x,x)=\sum_k\frac{1}{\sqrt{2k_0}}\frac{|\boldsymbol{k}|}{\varkappa_1}[U_3^*(k)\exp\{i(-\boldsymbol{k}\cdot\boldsymbol{x}+k_0x_0)\}$$

$$+U_3(k)\exp\{i(\boldsymbol{k}\cdot\boldsymbol{x}-k_0x_0)\}] \tag{38}$$

$$\boldsymbol{U}(x,x_0)=\sum_k\frac{1}{\sqrt{2k_0}}\left[\sum_{r=1.2}\left\{e_rU_r(k)+\frac{k_0\boldsymbol{k}}{\varkappa_1|\boldsymbol{k}|}U_3(k)\right\}\exp\{i(\boldsymbol{k}\cdot\boldsymbol{x}-k_0x_0)\}\right.$$

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* The term $igf\frac{Q}{2}\frac{d}{dt}\int v_0Vw'd\tau$ is missed there for the scalar meson field.

$$+\sum_{r=1,2}\left\{\boldsymbol{e}_r^* U_r^*(k)+\frac{k_0\boldsymbol{k}}{\varkappa_1|\boldsymbol{k}|}U_3^*(k)\right\}\exp\{i(-\boldsymbol{k}\cdot\boldsymbol{x}+k_0x_0)\}\Big], \tag{39}$$

$$\boldsymbol{F}(x,x_0)=\sum_k\frac{1}{\sqrt{2k_0}}\left(\frac{i}{\varkappa_1}\right)\Big[\left\{k_0\sum_{r=1,2}\boldsymbol{e}_rU_r(k)+\frac{\boldsymbol{k}\varkappa_1}{|\boldsymbol{k}|}U_3(k)\right\}\exp\{i(\boldsymbol{k}\cdot\boldsymbol{x}-k_0x_0)\}$$

$$-\left\{+k_0\sum_{r=1,2}\boldsymbol{e}_r^*U_r^*(k)+\frac{\boldsymbol{k}\varkappa_1}{|\boldsymbol{k}|}U_3^*(k)\right\}\exp\{i(-\boldsymbol{k}\cdot\boldsymbol{x}+k_0x_0)\}\Big], \tag{40}$$

$$\boldsymbol{G}(x,x_0)=\sum_k\frac{1}{\sqrt{2k_0}}\left(\frac{i}{\varkappa_1}\right)[\{-\sum_{r=1,2}[\boldsymbol{e}_r\times\boldsymbol{k}]U_r(k)\cdot\exp\,i(\boldsymbol{k}\cdot\boldsymbol{x}-k_0x_0)\}$$

$$+\{\sum_{r=1,2}[\boldsymbol{e}_r^*\times\boldsymbol{k}]U_r^*(k)\}\exp\{i(-\boldsymbol{k}\cdot\boldsymbol{x}+k_0x_0)\}], \tag{41}$$

where

$$N(k)=\sum_{r=1}^{3}U_r^*(k)\,U_r(k). \tag{42}$$

Hereafter, we shall quote the van Hove's results for the vector meson theory.

The straightforward calculation gives

$$e^{iS'}H(1)\,e^{-iS'}=H_N+H_{M_1}+H_{1.dir.}+H_1'' \tag{43}$$

$$H_N=(27\cdot1),\quad H_{M_1}=(28\cdot2),\quad H_{1.dir.}=(28\cdot4), \tag{44}$$

$$H_1''=\sum\Psi_{n_1n_2}\chi_{n_1}^{(1)*}\chi_{n_2}^{(2)*}(H_{v.}+H_{p.v.}+H_m^1)\chi_{n_1}^{(1)}\chi_{n_2}^{(2)}\Phi_{n_1n_2}, \tag{45}$$

$$H_{v.}=\text{van Hove's equ. (24)}, \tag{46}$$

$$H_{p.v.}=g_1'^2[(\sigma^{(1)}\cdot\sigma^{(2)})+\frac{1}{\varkappa_1^2}\{k^2\rho_1^{(2)}\rho_1^{(2)}+(\sigma^{(1)}\cdot\boldsymbol{k})(\sigma^{(2)}\cdot\boldsymbol{k})\}]$$

$$+\left(\frac{g_2'}{\varkappa_1}\right)^2[k_0^2\rho_3^{(1)}\rho_3^{(2)}(\sigma^{(1)}\cdot\sigma^{(2)})-\rho_3^{(2)}\rho_3^{(2)}(\sigma^{(1)}\cdot\boldsymbol{k})(\sigma^{(2)}\cdot\boldsymbol{k})$$

$$+k^2(\gamma^{(1)}\cdot\gamma^{(2)})-(\gamma^{(1)}\cdot\boldsymbol{k})(\gamma^{(2)}\cdot\boldsymbol{k})]$$

$$+i\left(\frac{g_1'g_2'}{\varkappa_1}\right)[(\boldsymbol{k}\cdot[\gamma^{(2)}\times\sigma^{(1)}])-(\boldsymbol{k}\cdot[\gamma^{(1)}\times\sigma^{(2)}])$$

$$-\rho_1^{(1)}\rho_3^{(2)}(\sigma^{(2)}\cdot\boldsymbol{k})+\rho_1^{(2)}\rho_3^{(1)}(\sigma^{(1)}\cdot\boldsymbol{k})], \tag{47}$$

$$H_m^1=g_1g_1'[(\alpha^{(2)}\cdot\sigma^{)1)})+(\alpha^{(1)}\cdot\sigma^{(2)})+\frac{1}{\varkappa_1^2}\{(\alpha^{(1)}\cdot\boldsymbol{k})(\sigma^{(2)}\cdot\boldsymbol{k})$$

$$+(\alpha^{(2)}\cdot\boldsymbol{k})(\sigma^{(1)}\cdot\boldsymbol{k})+k^2(\rho_1^{(2)}+\rho_1^{(1)})\}]$$

$$+g_2g_2'[\rho_3^{(1)}(\gamma^{(2)}\cdot\sigma^{(1)})+\rho_3^{(2)}(\gamma^{(1)}\cdot\sigma^{(2)})]$$

$$+i\left(\frac{g_1g_2'}{\varkappa_1}\right)[(\boldsymbol{k}\cdot[\gamma^{(2)}\times\alpha^{(1)}])-(\boldsymbol{k}\cdot[\gamma^{(1)}\times\alpha^{(2)}])+\rho_3^{(1)}(\sigma^{(1)}\cdot\boldsymbol{k})-\rho_3^{(2)}(\sigma^{(2)}\cdot\boldsymbol{k})]$$

$$+i\left(\frac{g_1'g_2}{\varkappa_1}\right)[(\rho_3^{(1)}+\rho_3^{(2)})(\boldsymbol{k}\cdot[\sigma^{(1)}\times\sigma^{(2)}])-\rho^{(1)}(\gamma^{(2)}\cdot\boldsymbol{k})+\rho_1^{(2)}(\gamma^{(1)}\cdot\boldsymbol{k})], \tag{48}$$

where $H_{v.}$, $H_{p.v.}$ and H_m^1 have the evident means as explained in the former case.

It is convenient for us to modify above formulas by the next relations.

$$\begin{cases} \boldsymbol{k}=\boldsymbol{P}_{m_1}-\boldsymbol{P}_{n_1}=\boldsymbol{P}_{n_2}-\boldsymbol{P}_{m_2}, \quad E_{m_1}-E_{m_2}=E_{n_1}-E_{n_2}=0, \\ (\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{k})=\rho_1^{(1)}(E_{m_1}-E_{n_1})+2iM\rho_2^{(1)}, \\ (\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{k})=\rho_1^{(2)}(E_{n_1}-E_{m_1})-2iM\rho_2^{(2)}. \end{cases} \tag{49}$$

Then the final results are

$$H_{v.}=\text{van Hove's equ. (25)}, \tag{50}$$

$$H_{p.v.}=\frac{1}{(E_{m_1}-E_{n_1})^2-k_0^2}[H_{p.v.}(E^0)+H_{p.v.}(E^1)+H_{p.v.}(E^2)]+H_{p.v.dir.}, \tag{51}$$

$$\begin{aligned} H_{p.v.}(E^0)=&g_1'^2[(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})-\rho_1^{(1)}\rho_1^{(2)}+\frac{4M^2}{\varkappa_1^2}\rho_2^{(1)}\rho_2^{(2)}] \\ &-\left(\frac{g_2'}{\varkappa_1}\right)^2[4M^2+\varkappa_1^2(\boldsymbol{\gamma}^{(1)}\cdot\boldsymbol{\gamma}^{(2)})]-\left(\frac{ig_1'g_2'}{\varkappa_1}\right)8M, \end{aligned} \tag{52}$$

$$\begin{aligned} H_{p.v.}(E^1)=&-g_1'^2\left[\frac{2iM}{\varkappa_1^2}(\rho_2^{(1)}\rho_1^{(2)}+\rho_1^{(1)}\rho_2^{(2)})(E_{m_1}-E_{n_1})\right] \\ &+\left(\frac{g_2'}{\varkappa_1}\right)^2[2M(\rho_3^{(1)}+\rho_3^{(2)})(E_{m_1}+E_{n_1})] \\ &+\left(\frac{g_1'g_2'}{\varkappa_1}\right)[i(\rho_3^{(1)}+\rho_3^{(2)})\{2+\rho_1^{(1)}\rho_1^{(2)}+(\boldsymbol{\alpha}^{(1)}\cdot\boldsymbol{\alpha}^{(2)})\}(E_{m_1}+E_{n_1}) \\ &\qquad -i(\rho_3^{(1)}+\rho_3^{(2)})(E_{m_1}-E_{n_1})], \end{aligned} \tag{53}$$

$$\begin{aligned} H_{p.v.}(E^2)=&\left(\frac{g_2'}{\varkappa_1}\right)^2[(E_{m_1}-E_{n_1})^2\{\rho_3^{(1)}\rho_2^{(2)}(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})+(\boldsymbol{\gamma}^{(1)}\cdot\boldsymbol{\gamma}^{(2)})\} \\ &\qquad -(E_{m_1}+E_{n_1})^2(\rho_2^{(1)}\rho_2^{(2)}+\rho_3^{(1)}\rho_3^{(2)})], \end{aligned} \tag{54}$$

$$H_{p.v.dir.}=\left(\frac{g_1'}{\varkappa_1}\right)^2\rho_1^{(1)}\rho_1^{(2)}+\left(\frac{g_2'}{\varkappa_1}\right)^2\rho_3^{(1)}\rho_3^{(2)}(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})+\left(\frac{g_2'}{\varkappa_1}\right)^2(\boldsymbol{\gamma}^{(1)}\cdot\boldsymbol{\gamma}^{(2)}), \tag{55}$$

$$H_m^1=\frac{1}{(E_{m_1}-E_{n_1})^2-k_0^2}[H_m^1(E^0)+H_m^1(E')]+H^1_{m.\ dir.}, \tag{56}$$

$$\begin{aligned} H_m^1(E^0)=&g_1g_1'[(\boldsymbol{\alpha}^{(2)}\cdot\boldsymbol{\sigma}^{(1)})+(\boldsymbol{\alpha}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})-(\rho_1^{(1)}+\rho_1^{(2)})] \\ &+g_2g_2'[\rho_3^{(2)}\rho_1^{(1)}(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})+\rho_3^{(1)}\rho_1^{(2)}(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})] \\ &-\left(\frac{g_1g_2'}{\varkappa_1}\right)[2M(\rho_2^{(1)}\rho_3^{(2)}+\rho_3^{(1)}\rho_2^{(2)})]-\left(\frac{g_1'g_2}{\varkappa_1}\right)[4M(\rho_1^{(1)}+\rho_1^{(2)})], \end{aligned} \tag{57}$$

$$H_m^1(E^1)=-g_1g_1'\left[\frac{2iM}{\varkappa_1^2}(E_{m_1}-E_{n_1})(\rho_2^{(1)}+\rho_2^{(2)})\right]$$

$$-\left(\frac{g_1 g_2'}{\varkappa_1}\right)[(E_{m_1}-E_{n_1})(\rho_2^{(1)}+\rho_2^{(2)})\{(\boldsymbol{\alpha}^{(1)}\cdot\boldsymbol{\alpha}^{(2)})-2\}]$$

$$+\left(\frac{g_1' g_2}{\varkappa_1}\right)[(E_{m_1}+E_{n_1})(\rho_3^{(1)}\rho_1^{(2)}+\rho_3^{(2)}\rho_1^{(1)})\{(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})+1\}$$

$$-i\{2(E_{m_1}+E_{n_1})+(E_{m_1}-E_{n_1})\}(\rho_2^{(1)}+\rho_2^{(2)})], \tag{58}$$

$$H^1_{m,\,dir.}=\left(\frac{g_1 g_1'}{\varkappa_1^2}\right)(\rho_1^{(1)}+\rho_1^{(2)}). \tag{59}$$

Now, with respect to the contact interaction term, it is impossible to obtain the unambiguous second order interaction for the spin 1 field. In fact, we can show that there is a term which has no counter term to be cancelled in the initial Hamiltonian. For example, the contact term $\left(\frac{g_2'}{\varkappa_1}\right)^2(\gamma^{(1)}\cdot\gamma^{(2)})$ which arises from (47) is in (55), while there is no term in (28·4) corresponding to it. Thus we should like to conclude that the spin 1 field is less successful than the spin 0 one. It is, however, necessary for us to note the following point. Although we have got discontent answers for the spin 1 field, we can nevertheless find that if we take only the usual vector meson theory, the contact term could be dropped out in a relativistically invariant way. Because it will be given in the van Hove's equ. (25) as follows:

$$H_{v.}=\frac{1}{(E_{m_1}-E_{n_1})^2-k_0^2}[H_{v.}(E^0)+H_{v.}(E^1)+H_{v.}(E^2)]+H_{v.dir}, \tag{60}$$

$$H_{v.dir.}=-H^{v.}_{dir.}-\left(\frac{g_2}{\varkappa_1}\right)^2\rho_3^{(1)}\rho_3^{(2)}. \tag{61}$$

The last term is a scalar, therefore adding the counter term of it to the initial Lagrangian we are able to eliminate all contact interaction terms.

§ 4. Nuclear potentials for the spin 0 field.

By the above considerations we have to examine more closely the spin 0 field. In (33) there are Dirac matrices such as ρ_1, ρ_2, and ρ_3.

We shall first intend to reduce them to $\boldsymbol{\sigma}$ by means of the Dirac equation for the two body problem which have been used by Araki.[7]

Now, we may go over by a ordinary method to the configuration space, and carry out the following computations in the non-relativistic and static limit. In this approximation (33) leads to

$$W^0=-\sum_k[f_3'^2\rho_2^{(1)}\rho_2^{(2)}+f_1^2\rho_3^{(1)}\rho_3^{(2)}+f_1f_3'(\rho_2^{(1)}\rho_3^{(2)}+\rho_2^{(2)}\rho_3^{(1)})]\frac{e^{i\boldsymbol{k}\cdot\boldsymbol{r}}}{k^2+\varkappa_0^2}. \tag{62}$$

The summation is replaced by integral and this gives rise to the Yukawa potential. Then, the Dirac equation is

$$H\Psi=E\Psi, \tag{63}$$

$$H=(\rho_1^{(1)}\sigma^{(1)}\boldsymbol{P}_1+\rho_1^{(2)}\sigma^{(2)}\boldsymbol{P}_2+M\rho_3^{(1)}+M\rho_3^{(2)}+W^0), \tag{64}$$

$$W^0=-(f_3^{\prime 2}\rho_2^{(1)}\rho_2^{(2)}+f_1^{\,2}\rho_3^{(1)}\rho_3^{(2)}+f_1f_3'(\rho_2^{(1)}\rho_3^{(2)}+\rho_2^{(2)}\rho_3^{(1)})\,\phi(r), \tag{65}$$

$$\phi(r)=\frac{1}{4\pi}\,\frac{1}{r}\exp(-\varkappa_0 r). \tag{66}$$

We expand Ψ in a linear combination of products of ψ and χ:

$$\Psi=\psi^{++}\chi_1^+\chi_2^++\psi^{+-}\chi_1^+\chi_2^-+\psi^{-+}\chi_1^-\chi_2^++\psi^{--}\chi_1^-\chi_2^-, \tag{67}$$

where χ^+ and χ^- are the normalized eigenfunctions of ρ_3 belonging to its eigenvalues $+1$ and -1 respectively, and ρ_1, ρ_2 and ρ_3 have the following properties for χ^+ and χ^-,

$$\begin{cases} \rho_1\chi^+=\chi^-, & \rho_2\chi^+=i\chi^-, & \rho_3\chi^+=\chi^+, \\ \rho_1\chi^-=\chi^+, & \rho_2\chi^-=-i\chi^+, & \rho_3\chi^-=-\chi^-. \end{cases} \tag{68}$$

Substituting (67) into (63), we have a system of simultaneous equation for ψ^{++}, ψ^{+-}, $\psi^{-+}$and ψ^{--}.

$$\left\{\begin{aligned} &(E-\phi f_1^{\,2}+\phi f_3^{\prime 2})(\psi^{+-}+\psi^{-+})=(\sigma^{(1)}\cdot\boldsymbol{P}_1+\sigma^{(2)}\cdot\boldsymbol{P}_2-2i\phi f_1f_3')(\psi^{++}+\psi^{--}), && (69)\\ &(E-\phi f_1^{\,2}-\phi f_3^{\prime 2})(\psi^{+-}-\psi^{-+})=-(\sigma^{(1)}\cdot\boldsymbol{P}_1+\sigma^{(2)}\cdot\boldsymbol{P}_2)(\psi^{++}-\psi^{--}), && (70)\\ &(E'-\phi f_3^{\prime 3}+\phi f_1^{\,2})\psi^{++}+(E''-\phi f_3^{\prime 2}+\phi f_1^{\,2})\psi^{--}=(\sigma^{(1)}\cdot\boldsymbol{P}_1+\sigma^{(2)}\cdot\boldsymbol{P}_2\\ &\qquad\qquad +2i\phi f_3'f_1)(\psi^{+-}+\psi^{-+}), && (71)\\ &-(E'+\phi f_3^{\prime 2}+\phi f_1^{\,2})\psi^{++}+(E''+\phi f_3^{\prime 2}+\phi f_1^{\,2})\psi^{--}=(\sigma^{(1)}\cdot\boldsymbol{P}_1-\sigma^{(2)}\cdot\boldsymbol{P}_2)(\psi^{+-}-\psi^{-+}), && (72) \end{aligned}\right.$$

where $\phi=\phi(r),\qquad E'=E-2M,\qquad E''=E+2M.$ (73)

From (71) we obtain

$$\begin{aligned} E'\psi^{++}=-\phi f_1^{\,2}\psi^{++}+[\phi f_3^{\prime 2}\psi^{++}-(E''-\phi f_3^{\prime 2}+\phi f_1^{\,2})\psi^{--}\\ +(\sigma^{(1)}\cdot\boldsymbol{P}_1+\sigma^{(2)}\cdot\boldsymbol{P}+2i\phi f_3'f_1)(\psi^{+-}+\psi^{-+})] \end{aligned} \tag{74}$$

Utilizing the remaining relations (74) is transformed into the equation involving only $\boldsymbol{\sigma}$ on the both sides.

$$\begin{aligned} E'\psi^{++}=\Big[\frac{1}{E}(P_1^{\,2}+P_2^{\,2})-\phi f_1^{\,2}-\frac{1}{E^2}\{(\sigma^{(1)}\cdot\boldsymbol{P}_1)\phi(\sigma^{(2)}\cdot\boldsymbol{P}_2)f_3^{\prime 2}+(\sigma^{(2)}\cdot\boldsymbol{P}_2)\phi(\sigma^{(1)}\cdot\boldsymbol{P}_1)f_3^{\prime 2}\\ -(\sigma^{(1)}\cdot\boldsymbol{P}_1)\phi(\sigma^{(1)}\cdot\boldsymbol{P}_1)f_1^{\,2}-(\sigma^{(2)}\cdot\boldsymbol{P}_2)\phi(\sigma^{(2)}\cdot\boldsymbol{P}_2)f_1^{\,2}\}\Big]\psi^{++}\\ +\Big[\Big\{\phi f_3^{\prime 2}+\frac{2}{(E-\phi f_1^{\,2}+\phi f_3^{\prime 2})}(\sigma^{(1)}\cdot\boldsymbol{P}_1)(\sigma^{(2)}\cdot\boldsymbol{P}_2)\Big\}\frac{(E'+\phi f_3^{\prime 2}+\phi f_1^{\,2})}{(E''+\phi f_1^{\,2}+\phi f_3^{\prime 2})} \end{aligned}$$

$$+\frac{\phi f_3'^2}{(E''+\phi f_1^2+\phi f_3'^2)}\{(\sigma^{(1)}\cdot\boldsymbol{P}_1)-(\sigma^{(2)}\cdot\boldsymbol{P}_2)\}\frac{1}{E}\{(\sigma^{(2)}\cdot\boldsymbol{P}_2)-(\sigma^{(1)}\cdot\boldsymbol{P}_1)\}\Big]\psi^{++}, \tag{75}$$

where $P_1=|\boldsymbol{P}_1|$, and $P_2=|\boldsymbol{P}_2|$.

Thus in the approximation mentioned above (75) becomes

$$E'\psi^{++}=\Big[\frac{1}{2M}(P_1^2+P_2^2)-\phi f_1^2+\frac{1}{(2M)^2}(\sigma^{(1)}\cdot\boldsymbol{P}_1)(\sigma^{(2)}\cdot\boldsymbol{P}_2)(\phi f_2'^2+\phi f_1^2)\Big]\psi^{++}, \tag{76}$$

where $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ do not operate on ψ^{++}.

Remarking that

$$(\sigma^{(1)}\cdot\boldsymbol{P}_1)(\sigma^{(2)}\cdot\boldsymbol{P}_2)\frac{e^{-k_0r}}{r}=\varkappa_0^2\frac{e^{-k_0r}}{r}\Big\{\frac{1}{3}(\sigma^{(1)}\cdot\sigma^{(2)})$$
$$+\Big(\frac{3(\sigma^{(1)}\cdot\boldsymbol{r})(\sigma^{(2)}\cdot\boldsymbol{r})}{r^2}-(\sigma^{(1)}\cdot\sigma^{(2)})\Big)\Big(\frac{1}{3}+\frac{1}{\varkappa_0 r}+\frac{1}{\varkappa_0^2r^2}\Big)\Big\}, \tag{77}$$

we get the final result:

$$W^0=-\Big[f_1^2-\Big(\frac{\varkappa_0}{2M}\Big)^2\Big\{\Big(f_1'+\frac{2M}{\varkappa_0}f_2'\Big)^2+f_1^2\Big\}\Big\{\frac{1}{3}(\sigma^{(1)}\cdot\sigma^{(2)})+S\omega(r)\Big\}\Big]\phi(r), \tag{78}$$

in which

$$S=\Big(\frac{3(\sigma^{(1)}\cdot\boldsymbol{r})(\sigma^{(2)}\cdot\boldsymbol{r})}{r^2}-(\sigma^{(1)}\cdot\sigma^{(2)})\Big),\quad \omega(r)=\Big(\frac{1}{3}+\frac{1}{\varkappa_0 r}+\frac{1}{\varkappa_0^2r^2}\Big). \tag{79}$$

Hitherto it has been assumed that the meson fields are pure neutral, but when the symmetrical and charged theories are considered, the following operators must be introduced in the right side of (78).

$$T=(\tau^{(1)}\cdot\tau^{(2)})/2\ (\textit{Symmetrical}),\quad T=(\tau_1^{(1)}\tau_2^{(2)}+\tau_2^{(1)}\tau_1^{(2)})/2\ (\textit{Charged}). \tag{80}$$

§ 5. Comprison with the Bethe's potential.

It should be noted that the so-called cross terms which have been in (62) vanish in (78) by virtue of our approximation. It is impossible to distinguish whether (78) is derived from the mixed model or the generalized interaction one. Further, it is obvious that if $f_1=0$ (or if $f_1'=f_2'=0$), (78) gives the potential of the pseudoscalar meson field (or the scalar meson field). For simplicity we assume the relations between coupling constants:

$$f_1=f_1'=f_2'=f.\quad \text{(Dimension of an electric charge !!)} \tag{81}$$

Then we shall write down the potentials for various theories.

Symmetrical theory.

Singlet state. $$W_s^s \sim -f^2 \phi(r). \qquad (82)$$

Triplet state. $$W_s^t \sim \frac{3}{2}\left(\frac{2}{3}f^2 - f^2\, S\omega(r)\right)\phi(r). \qquad (83)$$

Charged theory.

Singlet strte. $$W_c^s \sim -2f^2 \phi(r). \qquad (84)$$

Triplet state. $$W_c^t \sim \left(\frac{2}{3}f^2 - f^2\, S\omega(r)\right)\phi(r). \qquad (85)$$

On the other hand, Bethe's[8)] potentials by neutral vector meson theory are

Singlet state. $$W_B^s = -2g^2 \phi(r). \qquad (86)$$

Triplet state. $$W_B^t = \left(\frac{2}{3}g^2 - g^2\, S\omega(r)\right)\phi(r). \qquad (87)$$

Comparing W_s and W_c with W_B, we find that our potentials have the same character as Bethe's does.

§ 6. Concluding remark.

We have been shown that the expressions for the nuclear forces by employing the mixed meson theory or the spin 0 meson field are found to be in symmetrical and charged cases just the same as that obtained by Bethe for the neutral vector meson theory.

Although in this way the theory of nuclear forces can be brought into more satisfactory form, the well-known difficulties of the potential at the origin are inherited also by our theory. The singularities are, so to speak, the proper characters of the meson field, and thus a more profound modification of the theory is required in order to remove these difficulties.

In spite of the provisory character of our treatment, that is, nonrelativistic and static approximation, and relations between coupling constants, one should not forget that the result obtained above might have a few features in contrast to the method by using the ordinary perturbation theory. Firstly, contrary to the previous treatment the contact interaction terms do not appear in our case for the spin 0 field.

With respect to the mixed theory, various attempts[12)] have been put forward. For example, it has also been suggested by Yamasaki[13)] that the reasonable values of the deuteron problem could be obtained by mixing a scalar field with a pseudoscalar one. Nevertheless, the ambiguity of contact interaction terms still remained, for he carried out his computation by the customary perturbation method. Secondly, we should like to point out that our potential is in conformity with the expression derived from the relativistic formulation of the perturbation theory.

Apart from these problems, for the fixation of the choice between the two possible types of spin 0 fields (symmetrical or charged) responsible for the nuclear forces, a criterion is afforded by experiments. The results of recent ones seem quite cleary to indicate the presence of neutral π mesons. It seems, therefore, plausible to assume the symmetrical spin 0 field for the π meson. There are, however, some arguments for it, because according to Taketani, Nakamura and Sasaki,[14] it is demanded that in order to explain the angular distribution of the neutron-proton scattering, a neutral meson field must be added to a symmetrical one.

However it may be, it is a promising trial for the problem of nuclear forces to adopt the mixed field or the generalized interaction theory.

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Note on the Self-Energy and Self-Stress. I.

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1. Introductory Remarks

The problem of self-stress of an electron interacting with electromagnetic field was firstly investigated in quantum theory by Pais[1], who found that this does not vanish and has finite value amounting $\alpha/2\pi\cdot m$ (α: fine structure constant, m: mass of the electron). Then, observing the relation between energy and momentum of moving electron he concluded that the current theory is not yet fully satisfactory as to the stability of the electron itself and the description of it as relativistic particle. On the other hand, the theory recently developed by Tomonaga and Schwinger proceeded in the way of renormalizing the mass of electron. This procedure is to amalgamate the additional energy of an electron due to interaction with other fields into the original mass by a canonical transformation. The reason for using such a canonical transformation is given from the view point that any subtraction must be performed by canonical transformation to maintain the Hamiltonian from which we started still to be Hamiltonian after the subtraction.

Thus, the fundamental requirements of current theory are of two sorts: the one is that the self-stress of any particle is to be zero to secure proper transformation relation of energy and momentum as particle and to stabilize the particle. The general transformation properties of energy and momentum tensor are as follows: ($\hbar=c=1$)

$$E=-\int T_{44}dV=-\frac{\int\mathring{T}_{44}\,d\mathring{V}-\beta^2\int\mathring{T}_{11}\,d\mathring{V}}{\sqrt{1-\beta^2}}$$

$$G=-i\int T_{41}dV=-\frac{\int\mathring{T}_{44}\,d\mathring{V}-\int\mathring{T}_{11}\,d\mathring{V}}{\sqrt{1-\beta^2}} \tag{1}$$

where $\circ$ indicates the rest system of the particle.

The other is that the additional term in $-\int\mathring{T}_{44}\,d\mathring{V}$ due to the interaction with other fields is to be renormalized by a canonical transformation into the theoretical mass.

Now, proceeding along these lines, we found that the second requirement is not independent to the first: i.e. the renormalization of mass by a canonical transformation does modify the self-stress, in general. Among various ordinary particle up to spin 1, only spin 1/2 particle satisfies above two requirements simultaneously, i.e. if the self-stress is made vanish by some devices such as f-field theory[2], then the additional energy can be amalgamated into original mass by a canonical transformation without affecting the self-stress. The particle with spin 0 or 1, say, so-called mesons, cannot be pictured as particle with mass $\mu+\delta\mu$ by a canonical transformation, even if the self-stress vanishes and (1) takes correct form as particle.

These states of affairs can be shown as follows: as a first example, we take the system of electromagnetic field and the electron: the canonical energy-momentum tensor of this system in Heisenberg representation is given by the following:

$$\mathring{T}_{\mu\nu}=\frac{\partial\mathring{\phi}_\alpha}{\partial x_\mu}\frac{\partial\mathring{\phi}_\alpha}{\partial x_\nu}-\frac{1}{2}\delta_{\mu\nu}\frac{\partial\mathring{\phi}_\alpha}{\partial x_\rho}\frac{\partial\mathring{\phi}_\alpha}{\partial x_\rho}+\frac{1}{2}\left(\mathring{\psi}^\dagger\gamma^\mu\frac{\partial\mathring{\psi}}{\partial x_\nu}-\frac{\partial\mathring{\psi}^\dagger}{\partial x_\nu}\gamma^\mu\mathring{\psi}\right) \tag{1·2}$$

where ° indicate that the quantity belongs to Heisenberg representation and $\mathring{\phi}_\alpha$ is four-potential of radiation field. To bring this expression into interaction representation, we seould perform the following canonial transformation[2]:

$$U[\sigma]\,\Phi_{\text{Heisenberg}}=\Phi[\sigma]\,;\quad i\frac{\delta\Phi[\sigma]}{\delta\sigma(x)}=H(x)\,\Phi[\sigma]$$

$$i\frac{\delta U[\sigma]}{\delta\sigma(x)}=H(x)\,U[\sigma]\,;\quad H(x)=ie\mathring{\psi}^\dagger(x)\gamma^\rho\mathring{\psi}(x)\mathring{\phi}_\rho(x) \tag{1·3}$$

Then, writing $U[\sigma]\mathring{\phi}_\alpha U^{-1}[\sigma]=\phi_\alpha$, etc., we obtain as energy-momentum tensor in interaction representation:

$$T_{\mu\nu}=U[\sigma]\,\mathring{T}_{\mu\nu}U^{-1}[\sigma]=\frac{\partial\phi_\alpha}{\partial x_\mu}\frac{\partial\phi_\alpha}{\partial x_\nu}-\frac{1}{2}\delta_{\mu\nu}\frac{\partial\phi_\alpha}{\partial x_\rho}\frac{\partial\phi_\alpha}{\partial x_\rho}$$

$$+\frac{1}{2}\left(\psi^\dagger\gamma^\mu\frac{\partial\psi}{\partial x_\nu}-\frac{\partial\psi^\dagger}{\partial x_\nu}\gamma^\mu\psi\right)+\frac{ie}{2}\psi^\dagger(\gamma^\mu(\gamma^\lambda N_\lambda)\gamma^\rho+\gamma^\rho(\gamma^\lambda N_\lambda)\gamma^\mu)\psi\cdot\phi_\rho\cdot N_\nu \tag{1·4}$$

where N_λ is unit normal vector of surface $\sigma(-(N_\lambda N_\lambda)=1)$.

Here, the first requirement says that $1-1$ component of this is to be zero. If it is not so, one should use some device as f-field theory to make it vanish. Then, to renormalize the electron mass by an amount δm, which came from the interaction with radiation field and also probably from the device used to remove $1-1$ component, we should make following canocical transformation in accordance with second requirement[4]:

$$\Phi[\sigma]=U[\sigma]\,\Phi_{\delta m}[\sigma]\,;\quad i\frac{\delta\Phi_{\delta m}[\sigma]}{\delta\sigma(x)}=(H(x)-\delta m\psi^\dagger(x)\psi(x))\,\Phi_{\delta m}[\sigma]$$

$$i\frac{\delta U[\sigma]}{\delta\sigma(x)}=\delta m\psi^\dagger(x)\psi(x)U[\sigma] \tag{1·5}$$

Denoting $U^{-1}[\sigma]F(x)U[\sigma]$ as $\boldsymbol{F}(x)$, we obtain:

$$\boldsymbol{T}_{\mu\nu}=U^{-1}[\sigma]\,T_{\mu\nu}U[\sigma]=\frac{\partial\phi_\alpha}{\partial x_\mu}\,\frac{\partial\phi_\alpha}{\partial x_\nu}-\frac{1}{2}\,\delta_{\mu\nu}\,\frac{\partial\phi_\alpha}{\partial x_\rho}\,\frac{\partial\phi_\rho}{\partial x_\rho}+\frac{1}{2}\left(\boldsymbol{\psi}^\dagger\gamma^\mu\,\frac{\partial\boldsymbol{\psi}}{\partial x_\nu}-\frac{\partial\boldsymbol{\psi}^\dagger}{\partial x_\nu}\,\gamma^\mu\boldsymbol{\psi}\right)$$
$$+\frac{ie}{2}\,\boldsymbol{\psi}^\dagger(\gamma^\mu(\gamma^\lambda N_\lambda)\gamma^\rho+\gamma^\rho(\gamma^\lambda N_\lambda)\gamma^\mu)\,\boldsymbol{\psi}\cdot\phi_\rho\cdot N_\nu-N_\mu N_\nu\cdot\delta\boldsymbol{m}\,\boldsymbol{\psi}^\dagger\boldsymbol{\psi} \qquad (1\cdot6)$$

By this transformation, of course, the equation of motion of the electron becomes as that of mass $m+\delta m=\boldsymbol{m}$: $(\delta\boldsymbol{m}=\delta m+o(\delta m^2))$

$$\left(\gamma^\sigma\frac{\partial}{\partial x_\sigma}+\boldsymbol{m}\right)\boldsymbol{\psi}=0 \qquad (1\cdot7)$$

Now, $(1\cdot6)$ shows that $\boldsymbol{T}_{44}$ just gives the value as an electron with mass $\boldsymbol{m}$, and since the subtractive term only affects 4—4 component:

$$<\boldsymbol{T}_{44}>=<T_{44}>_{m=\boldsymbol{m}}+\delta\boldsymbol{m}<\psi^\dagger\psi>$$
$$<\boldsymbol{T}_{11}>=<T_{11}>_{m=\boldsymbol{m}} \qquad (1\cdot8)$$

So that, if the self-stress has been vanished (by f-field, for example) the electron becomes stable and we have correct energy and momentum relation as a particle with mass $m+\delta m$, since the additional energy to 4—4 component can be amalgamated into original mass without affecting the self-stress.

The different situation occures for spin 0 or 1 particle. We treat as the second example, the scalar meson interacting with electron (f-field). The canonical energy-momentum tensor in interaction representation is as follows:

$$T_{\mu\nu}=\frac{\partial\phi}{\partial x_\mu}\,\frac{\partial\phi}{\partial x_\nu}-\frac{1}{2}\,\delta_{\mu\nu}\left(\frac{\partial\phi}{\partial x_\rho}\,\frac{\partial\phi}{\partial x_\rho}+\mu^2\phi\phi\right)$$
$$+\frac{1}{2}\left(\psi^\dagger\gamma^\mu\,\frac{\partial\psi}{\partial x_\nu}-\frac{\partial\psi^\dagger}{\partial x_\nu}\,\gamma^\mu\psi\right)+fN_\mu N_\nu\psi^\dagger\psi\phi \qquad (1\cdot9)$$

where ϕ is meson variable, μ is meson mass. Here, also, according to the first requirement we should obtain vanishing 1—1 component for meson. Now, assuming this requirement is fulfilled by some device, to amalgamate the mass correction of meson due to virtual electron-positron pairs, we should transform the state functional as follows:

$$\Phi[\sigma]=U[\sigma]\Phi_{\delta\mu}[\sigma]; \quad i\,\frac{\delta\Phi_{\delta\mu}[\sigma]}{\delta\sigma(x)}=(H(x)-\mu\delta\mu\phi(x)\,\phi(x))\,\Phi_{\delta\mu}[\sigma]$$
$$i\,\frac{\delta U[\sigma]}{\delta\sigma(x)}=\mu\delta\mu\phi(x)\,\phi(x)\,U[\sigma]; \quad H(x)=f\psi^\dagger(x)\,\phi(x) \qquad (1\cdot10)$$

By this transformation, the equation of motion of meson becomes:

$$\left(\frac{\partial^2}{\partial x_\rho{}^2}-(\mu^2+2\mu\delta\mu)\right)\phi=\left(\frac{\partial^2}{\partial x_\rho{}^2}-(\mu+\delta\mu)^2\right)\phi=0 \qquad (1\cdot11)$$

regarding $\delta\mu$ as small. Energy-momentum tensor becomes as follows:

$$\boldsymbol{T}_{\mu\nu}=U^{-1}[\sigma]\,T_{\mu\nu}U[\sigma]=\frac{\partial\phi}{\partial x_\mu}\frac{\partial\phi}{\partial x_\nu}-\frac{1}{2}\delta_{\mu\nu}\left(\frac{\partial\phi}{\partial x_\rho}\frac{\partial\phi}{\partial x_\rho}+\mu^2\phi\,\phi\right)$$

$$+\frac{1}{2}\left(\psi^\dagger\gamma^\mu\frac{\partial\psi}{\partial x_\nu}-\frac{\partial\psi^\dagger}{\partial x_\nu}\gamma_\mu\psi\right)+fN_\mu N_\nu\psi^*\psi\cdot\boldsymbol{\phi}+\delta_{\mu\nu}\cdot\mu\,\delta_\mu\boldsymbol{\phi}\boldsymbol{\phi} \qquad (1\cdot12)$$

where $\boldsymbol{\mu}=\mu+\delta\mu$. $(\boldsymbol{\delta\mu}=\delta\mu+\alpha(\delta\mu^2))$.

Thus, this expression shows that the renormalization of additional mass by a canonical transformation can make $\boldsymbol{T}_{44}$ as that of meson with mass $\boldsymbol{\mu}$, but moreover modifies 1—1 componet by just the subtracted amount:

$$<\boldsymbol{T}_{44}>=<T_{44}>_{\mu=\boldsymbol{\mu}}+\mu\,\delta\mu<\phi\phi>$$

$$<\boldsymbol{T}_{11}>=<T_{11}>_{\mu-\boldsymbol{\mu}}+\mu\,\delta\mu<\phi\phi> \qquad (1\cdot13)$$

So that, if the self-stress has been vanished before transformation in accordance with first requirement, $-\int\mathring{T}_{11}d\mathring{V}=0$, which implies that the energy and momentum of meson has correct relation as particle, (1·13) shows that the meson becomes unstable and its energy-momentum relation as particle is destroyed after the renormalization procedure.

Thus, we arrived at the conclusion that the two requirements of current theory cannot be satisfied simaltaneously in this case: the vanishment of the self-stress, i.e., stability condition of meson is not independent with the mass renormalization by canonical transformation.

From above considerations, it seems that between spin 1/2 particle and spin 0 or 1 particle there is essential difference, the former can be pictured consistently as stable relativistic particle, but the latter cannot, along the self-consistent subtraction method of Tomonaga-Schwinger.[5] Here, it is to be remarked that as we will show in the following paragraphs the mass zero meson or photon has zero self-energy and self-stress, if the expressions of these quantities are adequately regularized, and only these can be pictured as stable relativistic particle among spin 0 or 1 particles.

If, the so called mesons in cosmic-rays are to be regarded as relativistic particle, they are to be spin 1/2, or otherwise, as a theoretical description of them, we should proceed another way from Tomonaga-Schwinger theory to make vanish both self-stress and-energy simaltaneously. The simaltaneous vanishment of these secures the particle aspect is obvious.

We discussed in the above the consistency of the two general requirements ot Tomonaga-Schwinger theory. Next, we want to tatch briefly to the realistic subtraction procedure. In this realistic procedure, different from the subtraction by the canonical transformation, one can subtract self-energy and self-stress independently by mixing enormalous number and sorts or fields, if one wants, and one can give the particle aspect to meson by making to vanish self-energy and self-stress simaltaneously. But since this point of view is realistic one, the system is to be closed: the another problem arises as to this closure of the system, before finding the closed system one should try and try, there seems to be no guiding principle to find this closed system at hand.

The situation be it so, we want to survey actual behavior of self-energy and self-stress in the following papragraphs, postponing the conclusion to the last paragraph.

In the course of calculations, we found general relation between $<\bar{T}_{11}>$ and $<\bar{T}_{44}>$, which includes Pais-Epstein relation as a special case. This will be discussed briefly at paragraph 7.*

2. Self-Stress of an Electron due to Scalar-Meson Field

Since the full account on this subject made by Pais[1] concerning f-field theory has not yet been appeared, we want to give summary of our calculation. As to the procedure of calcuation, it seems to us that the ordinary perturbation theory is only preferable one for the evaluation of expectation value of self-stress. Firstly, we transform (1·9) into Schroedinger representation:

$$\begin{aligned}\int T_{\mu\nu}dV=\sum_{\boldsymbol{k}}\frac{1}{2k_0}\Big[&\left\{\begin{matrix}\boldsymbol{k}\\ ik_0\end{matrix}\right\}_\mu\left\{\begin{matrix}\boldsymbol{k}\\ ik_0\end{matrix}\right\}_\nu(\phi(\boldsymbol{k})\,\phi^*(\boldsymbol{k})+\phi^*(\boldsymbol{k})\,\phi(\boldsymbol{k}))\\ -\frac{1}{2}\Big(&\left\{\begin{matrix}\boldsymbol{k}\\ ik_0\end{matrix}\right\}_\mu\left\{\begin{matrix}-\boldsymbol{k}\\ ik_0\end{matrix}\right\}_\nu+\left\{\begin{matrix}-\boldsymbol{k}\\ ik_0\end{matrix}\right\}_\mu\left\{\begin{matrix}\boldsymbol{k}\\ ik_0\end{matrix}\right\}+2\,\delta_{\mu\nu}k_0^2\Big)(\phi(\boldsymbol{k})\,\phi(-\boldsymbol{k})+\phi^*(\boldsymbol{k})\,\phi^*(-\boldsymbol{k}))\Big]\\ &+\sum a_{\boldsymbol{p}}^{*r}\,i\gamma^4\left\{\begin{matrix}\gamma\\ \gamma^4\end{matrix}\right\}_\mu\left\{\begin{matrix}\boldsymbol{p}\\ i\,\dfrac{E_{\boldsymbol{p}}^r+E_{\boldsymbol{p}}^s}{2}\end{matrix}\right\}_\nu a_{\boldsymbol{p}}^s\\ &-f\sum_{\boldsymbol{p}}\sum_{\boldsymbol{k}}a_{\boldsymbol{p}}^*\gamma^4(a_{\boldsymbol{p}-\boldsymbol{k}}\phi(\boldsymbol{k})+a_{\boldsymbol{p}+\boldsymbol{k}}\phi^*(\boldsymbol{k}))\frac{1}{\sqrt{2k_0}}\,\delta_{\mu4}\,\delta_{\nu4}\end{aligned}\qquad(2\cdot1)$$

where we took the surface σ as flat, $\phi^*(\boldsymbol{k})$, $\phi(\boldsymbol{k})$ are emission and absorption operators of meson with momentum $\boldsymbol{k}$, $k_0=\sqrt{|\boldsymbol{k}|^2+\mu^2}$ is meson energy, $a_{\boldsymbol{p}}^{r*}$, $a_{\boldsymbol{q}}^s$ are the creation and annihilation operators of electron in the state with energy $E_{\boldsymbol{p}}^r$ and $E_{\boldsymbol{q}}^s$ respectively, r, s designate the positive and negative energy states, and $\{\ \}_\mu$ means the component of a four-vector.

To obtain the expectation value of (2·1) for the state with one electron in state of momentum $\boldsymbol{p}_0$, one need to construct $(\Phi^\dagger\cdot\int T_{\mu\nu}dV\cdot\Phi)$ with perturbed state functional Ψ. If one confines his attention to second order approximation, one need to take Φ up to second order expansion, but from (2·1) it is clear that among second order terms only states with one electron, one pair of electron, no meson and one-electron, two mesons contribute to the expectation value. Thus, the effective part of perturbed state functional becomes as follows:

$$\begin{aligned}
\Phi=\Phi(\boldsymbol{p}_0^+,0^-,0^\sim)&+f\sum_{\boldsymbol{k}}\frac{\Phi(\boldsymbol{p}_0^+-\boldsymbol{k},0^-,\boldsymbol{k}^\sim)(\boldsymbol{p}_0^+-\boldsymbol{k}|\gamma^4|\boldsymbol{p}_0^+)}{\sqrt{2k_0}\,(E_{\boldsymbol{p}_0}-E_{\boldsymbol{p}_0-\boldsymbol{k}}-k_0)}\\
&+f\sum_{\boldsymbol{k}}\sum_{\boldsymbol{l}\neq\boldsymbol{p}_0}\frac{\Phi(\boldsymbol{p}_0^+,\boldsymbol{l}^+,\boldsymbol{l}^-+\boldsymbol{k},\boldsymbol{k}^\sim)(\boldsymbol{l}^+|\gamma^4|\boldsymbol{l}^-+\boldsymbol{k})}{\sqrt{2k_0}\,(-E_{\boldsymbol{l}}-E_{\boldsymbol{l}+\boldsymbol{k}}-k_0)}
\end{aligned}$$

$$\left.\begin{aligned}
&+f\sum_{\boldsymbol{k}}\sum_{\left(\substack{\boldsymbol{k}'\\=\boldsymbol{k}:\sqrt{2}\text{ times}}\right)}\\
&\times\frac{\Phi(\boldsymbol{p}_0^+-\boldsymbol{k}-\boldsymbol{k}',0^-,\boldsymbol{k}^\sim,\boldsymbol{k}'^\sim)(\boldsymbol{p}_0^+-\boldsymbol{k}-\boldsymbol{k}'|\gamma^4|\boldsymbol{p}_0^+-\boldsymbol{k})(\boldsymbol{p}_0^+-\boldsymbol{k}|\gamma^4|\boldsymbol{p}_0^+)}{\sqrt{2k_0}\sqrt{2k_0'}(E_{\boldsymbol{p}_0}-E_{\boldsymbol{p}_0-\boldsymbol{k}-\boldsymbol{k}'}-k_0-k_0')(E_{\boldsymbol{p}_0}-E_{\boldsymbol{p}_0-\boldsymbol{k}}-k_0)}\\
&+f^2\sum_{\boldsymbol{k}}\sum_{\boldsymbol{l}\neq\boldsymbol{p}_0}\frac{\Phi(\boldsymbol{p}_0^+,0^-,\boldsymbol{k}^\sim,-\boldsymbol{k}^\sim)(\boldsymbol{l}^-+\boldsymbol{k}|\gamma^4|\boldsymbol{l}^+)(\boldsymbol{l}^+|\gamma^4|\boldsymbol{l}^-+\boldsymbol{k})}{2k_0(-k_0-k_0)(-E_{\boldsymbol{l}}-E_{\boldsymbol{l}+\boldsymbol{k}}-k_0)}
\end{aligned}\right\}\text{(a)}$$

$$\left.\begin{aligned}
&+f^2\sum_{\boldsymbol{k}}\sum_{\boldsymbol{l}\neq\boldsymbol{p}_0-\boldsymbol{k}}\frac{\Phi(\boldsymbol{p}_0^+-\boldsymbol{k},\boldsymbol{l}^+,\boldsymbol{l}^--\boldsymbol{k},0^\sim)(\boldsymbol{l}^+|\gamma^4|\boldsymbol{l}^--\boldsymbol{k})(\boldsymbol{p}_0^+-\boldsymbol{k}|\gamma^4|\boldsymbol{p}_0^+)}{2k^0(E_{\boldsymbol{p}_0}-E_{\boldsymbol{p}_0-\boldsymbol{k}}-E_{\boldsymbol{l}}-E_{\boldsymbol{l}-\boldsymbol{k}})(E_{\boldsymbol{p}_0}-E_{\boldsymbol{p}_0-\boldsymbol{k}}-k_0)}\\
&+f^2\sum_{\boldsymbol{k}}\sum_{\boldsymbol{l}\neq\boldsymbol{p}_0,\boldsymbol{p}_0+\boldsymbol{k}}\frac{\Phi(\boldsymbol{p}_0^++\boldsymbol{k},\boldsymbol{l}^+,\boldsymbol{l}^-+\boldsymbol{k},0^\sim)(\boldsymbol{p}_0^-+\boldsymbol{k}|\gamma^4|\boldsymbol{p}_0^-)(\boldsymbol{l}^+|\gamma^4|\boldsymbol{l}^-+\boldsymbol{k})}{2k_0(E_{\boldsymbol{p}_0}-E_{\boldsymbol{p}_0+\boldsymbol{k}}-E_{\boldsymbol{l}}-E_{\boldsymbol{l}+\boldsymbol{k}})(-E_{\boldsymbol{l}}-E_{\boldsymbol{l}+\boldsymbol{k}}-k_0)}\\
&+f^2\sum_{\boldsymbol{k}}\sum_{\boldsymbol{l}\neq\boldsymbol{p}_0,\boldsymbol{p}_0-\boldsymbol{k}}\frac{\Phi(\boldsymbol{p}_0^+,\boldsymbol{l}^++\boldsymbol{k},\boldsymbol{l}^-+\boldsymbol{k},0^\sim)(\boldsymbol{l}^-+\boldsymbol{k}|\gamma^4|\boldsymbol{l}^+)(\boldsymbol{l}^+|\gamma^4|\boldsymbol{l}^-+\boldsymbol{k})}{2k_0(-E_{\boldsymbol{l}+\boldsymbol{k}}-E_{\boldsymbol{l}+\boldsymbol{k}})(-E_{\boldsymbol{l}}-E_{\boldsymbol{l}+\boldsymbol{k}}-k_0)}\\
&+f^2\sum_{\boldsymbol{k}}\sum_{\boldsymbol{l}\neq\boldsymbol{p}_0}\frac{\Phi(\boldsymbol{p}_0^+,\boldsymbol{l}^+,\boldsymbol{l}^-,0^\sim)(\boldsymbol{l}^-+\boldsymbol{k}|\gamma^4|\boldsymbol{l}^-)(\boldsymbol{l}^+|\gamma^4|\boldsymbol{l}^-+\boldsymbol{k})}{2k_0(-E_{\boldsymbol{l}}-E_{\boldsymbol{l}})(-E_{\boldsymbol{l}}-E_{\boldsymbol{l}+\boldsymbol{k}}-k_0)}
\end{aligned}\right\}\text{(b)}$$

$$\begin{aligned}
&-\frac{1}{2}f^2\sum_{\boldsymbol{k}}\Bigg[\frac{(\boldsymbol{p}_0^+|\gamma^4|\boldsymbol{p}_0^+-\boldsymbol{k})(\boldsymbol{p}_0^+-\boldsymbol{k}|\gamma^4|\boldsymbol{p}_0^+)}{2k_0(E_{\boldsymbol{p}_0}-E_{\boldsymbol{p}_0-\boldsymbol{k}}-k_0)^2}\\
&\qquad+\sum_{\boldsymbol{l}\neq\boldsymbol{p}_0}\frac{(\boldsymbol{l}^-+\boldsymbol{k}|\gamma^4|\boldsymbol{l}^+)(\boldsymbol{l}^+|\gamma^4|\boldsymbol{l}^-+\boldsymbol{k})}{2k_0(-E_{\boldsymbol{l}}-E_{\boldsymbol{l}+\boldsymbol{k}}-k_0)^2}\Bigg]\Phi(\boldsymbol{p}_0^+,0^-,0^\sim)\quad\text{(c)}\qquad(2\cdot2)
\end{aligned}$$

where $\Phi(\boldsymbol{p}_0^+,0^-,0^\sim)$ means the unperturbed state functional with one-electron, filled negative energy states and no meson, and so on. (a) are states with one-electron, two mesons and (b) are those with one-electron, one pair of electron, no meson and (c) is the renormalization of probability amplitude. Then, observing (2·1), it is sufficient to retain for (a) only those states in which mesons have opposite momentum and for (b) electron and hole have same momentum. Constructing $(\Phi^\dagger\cdot\int T_{\mu\nu}dV\cdot\Phi)$ and subtracting from this the value of vacuum, we obtain as additional term to $<\int T_{\mu\nu}dV>_0$:

$$\delta<\int T_{\mu\nu}dV>=$$

$$=f^2\sum_{\boldsymbol{k}}\frac{(\boldsymbol{p}_0^+|\gamma^4|\boldsymbol{p}_0^+-\boldsymbol{k})(\boldsymbol{p}_0^+-\boldsymbol{k}|\gamma^4|\boldsymbol{p}_0^+)}{2k_0(E_{\boldsymbol{p}_0}-E_{\boldsymbol{p}_0-\boldsymbol{k}}-k_0)^2}\Big[-\boldsymbol{p}_0^+|\begin{Bmatrix}\boldsymbol{\alpha}\\ i\end{Bmatrix}_\mu\begin{Bmatrix}\boldsymbol{p}_0\\ iE_{\boldsymbol{p}_0}\end{Bmatrix}_\nu|\boldsymbol{p}_0^+)$$

$$+(\boldsymbol{p}_0^+-\boldsymbol{k}|\begin{Bmatrix}\alpha\\ i\end{Bmatrix}_\mu\begin{Bmatrix}\boldsymbol{p}_0-\boldsymbol{k}\\ iE_{\boldsymbol{p}_0-\boldsymbol{k}}\end{Bmatrix}_\nu|\boldsymbol{p}_0^+-\boldsymbol{k})+\frac{1}{k_0}\begin{Bmatrix}\boldsymbol{k}\\ ik_0\end{Bmatrix}_\mu\begin{Bmatrix}\boldsymbol{k}\\ ik_0\end{Bmatrix}_\nu\Bigg]$$

$$-f^2\sum_{\boldsymbol{k}}\frac{(\boldsymbol{p}_0^-+\boldsymbol{k}|\gamma^4|\boldsymbol{p}_0^+)(\boldsymbol{p}_0^+|\gamma^4|\boldsymbol{p}_0^-+\boldsymbol{k})}{2k_0(-E_{\boldsymbol{p}_0}-E_{\boldsymbol{p}_0+\boldsymbol{k}}-k_0)}\Bigg[(\boldsymbol{p}_0^+|\begin{Bmatrix}\alpha\\ i\end{Bmatrix}_\mu\begin{Bmatrix}\boldsymbol{p}_0\\ iE_{\boldsymbol{p}_0}\end{Bmatrix}_\nu|\boldsymbol{p}_0^+)$$

$$+(\boldsymbol{p}_0^-+\boldsymbol{k}|\begin{Bmatrix}\alpha\\ i\end{Bmatrix}_\mu\begin{Bmatrix}-\boldsymbol{p}_0-\boldsymbol{k}\\ iE_{\boldsymbol{p}_0+\boldsymbol{k}}\end{Bmatrix}_\nu|\boldsymbol{p}_0^-+\boldsymbol{k})+\frac{1}{k_0}\begin{Bmatrix}\boldsymbol{k}\\ ik_0\end{Bmatrix}_\mu\begin{Bmatrix}\boldsymbol{k}\\ ik_0\end{Bmatrix}_\nu\Bigg]$$

$$-2f^2\sum_{\boldsymbol{k}}\left[\frac{(\boldsymbol{p}_0^+|\gamma^4|\boldsymbol{p}_0^+-\boldsymbol{k})(\boldsymbol{p}_0^+-\boldsymbol{k}|\gamma^4|\boldsymbol{p}_0^+)}{2k_0(E_{\boldsymbol{p}_0}-E_{\boldsymbol{p}_0-\boldsymbol{k}}-k_0)}-\frac{(\boldsymbol{p}_0^-+\boldsymbol{k}|\gamma^4|\boldsymbol{p}_0^+)(\boldsymbol{p}_0^+|\gamma^4|\boldsymbol{p}_0^-+\boldsymbol{k})}{2k_0(-E_{\boldsymbol{p}_0}-E_{\boldsymbol{p}_0+\boldsymbol{k}}-k_0)}\right]\delta_{\mu4}\delta_{\nu4}$$

$$+f^2\sum_{\boldsymbol{k}}\frac{(\boldsymbol{p}_0^+|\gamma^4|\boldsymbol{p}_0^--\boldsymbol{k})(\boldsymbol{p}_0^--\boldsymbol{k}|\begin{Bmatrix}\alpha\\ i\end{Bmatrix}_\mu\begin{Bmatrix}\boldsymbol{p}_0-\boldsymbol{k}\\ 0\end{Bmatrix}_\nu|\boldsymbol{p}_0^+-\boldsymbol{k})(\boldsymbol{p}_0^+-\boldsymbol{k}|\gamma^4|\boldsymbol{p}_0^+)}{4k_0E_{\boldsymbol{p}_0-\boldsymbol{k}}(E_{\boldsymbol{p}_0}-E_{\boldsymbol{p}_0-\boldsymbol{k}}-k_0)}$$

$$+\ll\text{transposed}\gg$$

$$+f^2\sum_{\boldsymbol{k}}\frac{(\boldsymbol{p}_0^+|\gamma^4|\boldsymbol{p}_0^-+\boldsymbol{k})(\boldsymbol{p}_0^-+\boldsymbol{k}|\begin{Bmatrix}\alpha\\ i\end{Bmatrix}_\mu\begin{Bmatrix}\boldsymbol{p}_0+\boldsymbol{k}\\ 0\end{Bmatrix}_\nu|\boldsymbol{p}_0^++\boldsymbol{k})(\boldsymbol{p}_0^++\boldsymbol{k}|\gamma^4|\boldsymbol{p}_0^+)}{4k_0E_{\boldsymbol{p}_0+\boldsymbol{k}}(-E_{\boldsymbol{p}_0}-E_{\boldsymbol{p}_0+\boldsymbol{k}}-k_0)}$$

$$+\ll\text{transposed}\gg$$

$$+f^2\sum_{\boldsymbol{k}}\left[\frac{(\boldsymbol{p}_0^+|\gamma^4|\boldsymbol{p}_0^+-\boldsymbol{k})(\boldsymbol{p}_0^+-\boldsymbol{k}|\gamma^4|\boldsymbol{p}_0)}{2k_0(E_{\boldsymbol{p}_0}+E_{\boldsymbol{p}_0-\boldsymbol{k}}-k_0)}-\frac{(\boldsymbol{p}_0^+|\gamma^4|\boldsymbol{p}_0^-+\boldsymbol{k})(\boldsymbol{p}_0^-+\boldsymbol{k}|\gamma^4|\boldsymbol{p}_0^-)}{2k_0(-E_{\boldsymbol{p}_0}-E_{\boldsymbol{p}_0+\boldsymbol{k}}-k_0)}\right].$$

$$\times\frac{(\boldsymbol{p}_0^-|\begin{Bmatrix}\alpha\\ i\end{Bmatrix}_\mu\begin{Bmatrix}\boldsymbol{p}_0\\ 0\end{Bmatrix}_\nu|\boldsymbol{p}_0^+)}{2E_{\boldsymbol{p}_0}}+\ll\text{transposed}\gg$$

$$+f^2\sum_{\boldsymbol{k}}\left[\frac{(\boldsymbol{p}_0^+|\gamma^4|\boldsymbol{p}_0^+-\boldsymbol{k})(\boldsymbol{p}_0^+-\boldsymbol{k}|\gamma^4|\boldsymbol{p}_0^+)}{4k_0^3(E_{\boldsymbol{p}_0}-E_{\boldsymbol{p}_0-\boldsymbol{k}}-k_0)}-\frac{(\boldsymbol{p}_0^-+\boldsymbol{k}|\gamma^4|\boldsymbol{p}_0^+)(\boldsymbol{p}_0^+|\gamma^4|\boldsymbol{p}_0^-+\boldsymbol{k})}{4k_0^3(-E_{\boldsymbol{p}_0}-E_{\boldsymbol{p}_0+\boldsymbol{k}}-k_0)}\right].$$

$$\times\left[\begin{Bmatrix}\boldsymbol{k}\\ ik_0\end{Bmatrix}_\mu\begin{Bmatrix}-\boldsymbol{k}\\ ik_0\end{Bmatrix}_\nu+\begin{Bmatrix}-\boldsymbol{k}\\ ik_0\end{Bmatrix}_\mu\begin{Bmatrix}\boldsymbol{k}\\ ik_0\end{Bmatrix}_\nu+2\delta_{\mu\nu}k_0^2\right] \qquad (2\cdot3)$$

From this expression, it follows immediately that $\delta<\int T_{44}dV>=-$ W$_{\text{self}}$ and $\delta<\int T_{41}dV>=0$. The latter is the general consequence of the conservation of momentum throughout initial, intermediate and final states. Thus, the self-momentum vanishes generally.

The expression for resting electron ($\boldsymbol{p}_0=0$) becomes:

$$\delta<\int\mathring{T}_{44}\,d\mathring{V}>=-f^2\sum_{\boldsymbol{k}}\frac{1}{4k_0E}\left[\frac{E+m}{m-E-k_0}-\frac{E-m}{-m-E-k_0}\right]$$

$$\delta<\int\mathring{T}_{11}\,dV>=f^2\sum_{\boldsymbol{k}}\frac{1}{3}\left[\frac{1}{4k_0E^2}+\frac{1}{4k_0^2E}\right]\left[\frac{E+m}{(m-E-k_0)^2}-\frac{E-m}{(-m-E-k_0)^2}\right]\cdot k^2$$

$$-f^2\sum_{\boldsymbol{k}}\frac{1}{3}\frac{m}{4k_0E^3}\left[\frac{1}{m-E-k_0}+\frac{1}{-m-E-k_0}\right]\cdot k^2$$

$$-f^2\sum_{\boldsymbol{k}}\frac{1}{3}\frac{1}{4k_0^3E}\left[\frac{E+m}{m-E-k_0}-\frac{E-m}{-m-E-k_0}\right]\cdot k^2$$
$$+f^2\sum_{\boldsymbol{k}}\frac{1}{4k_0E}\left[\frac{E+m}{m-E-k_0}-\frac{E-m}{-m-E-k_0}\right] \tag{2·4}$$

The last expression converges, and becomes for $\mu=0$ as follows:

$$\delta<\int\mathring{T}_{11}\,d\mathring{V}>_{\mu=0}=f^2\sum_{\boldsymbol{k}}\frac{1}{6Ek^2}m+f^2\sum_{\boldsymbol{k}}\frac{1}{12E^3}m-f^2\sum_{\boldsymbol{k}}\frac{2E+k}{6E(E+k)k^2}$$
$$=-\frac{(f^2/4\pi)}{4\pi}m \tag{2·5}$$

The same value is obtained for any value of μ by direct calculations.**

We should add here some notes as to the consequence lead by the vanishment of the self-momentum. If one solves the second equation of (1·1) for β and put this to the first equation, one obtains the energy E expressed by momentum $\boldsymbol{p}$ $(=\boldsymbol{G})$: then, writing $<-\int\mathring{T}_{11}d\mathring{V}>=m-\delta m$, and expanding to δm, one obtains as the expansion of E:

$$E=\sqrt{m^2+\boldsymbol{p}^2}+\frac{m}{\sqrt{m^2+\boldsymbol{p}^2}}\delta m+o(\delta m)^2 \tag{2·6}$$

It is to be noted that, in this expression, $<\int\mathring{T}_{11}d\mathring{V}>$ does not appear, and since we now have the result of vanishing self-momentum, which implies $\boldsymbol{p}=\boldsymbol{p}_0$ (momentum of the unperturced electron), and (2·6) becomes:

$$E=\sqrt{m^2+\boldsymbol{p}_0^2}+\frac{m}{\sqrt{m^2+\boldsymbol{p}_0^2}}\delta m=E_0+\frac{m}{E_0}\delta m \tag{2·7}$$

This shows that, even when the self-stress does exist, the additional energy written by the unperturbed energy and momentum has the form $\frac{m}{E_0}\delta m$. So that, though appearently this additional energy has the form $\frac{\partial E_0}{\partial m}\delta m$, this cannot be identified thoroughly as the mass correction term[6].

3. Self-Energy and Self-Stress of Scalar-Meson due to Electron-Positron Pairs

In the preceding section, we calculated the self-stress of electron due to the scalar meson field, then, inversely, we want to calculate self-stress of scalar meson due to the electron-positron pairs. This will serve to examine the closedness of the realistic subtraction procedure as C-meson or f-field theories.

As stated in I, the meson is only stable relativistic particle when it has no self-energy and self-stress, so that we want to survey under what condition these can be fulfilled in nowadays theory.

The perturbed state functional with one meson of momentum $\boldsymbol{k}$ in the unperturbed state which is effective in the calculation of second order modification of energy-momentum tensor is given by the following:

$$
\begin{aligned}
\Phi = {} & \Phi(0^+,0^-,\boldsymbol{k}^{\sim}) + f\sum_{\boldsymbol{l}} \frac{\Phi(\boldsymbol{l}^+ + \boldsymbol{k}, \boldsymbol{l}^-, 0^{\sim})(\boldsymbol{l}^+ + \boldsymbol{k}|\gamma^4|\boldsymbol{l}^-)}{\sqrt{2k_0}\,(-E_{\boldsymbol{l}} - E_{\boldsymbol{l}+\boldsymbol{k}} + k_0)} \\
& + f\sum_{\boldsymbol{l}} \sum_{\substack{\boldsymbol{k}' \\ (=\boldsymbol{k}:\,\sqrt{2}\text{ times})}} \frac{\Phi(\boldsymbol{l}^+, \boldsymbol{l}^- + \boldsymbol{k}, \boldsymbol{k}^{\sim}, \boldsymbol{k}^{\sim\prime})(\boldsymbol{l}^-|\gamma^4|\boldsymbol{l}^- + \boldsymbol{k}')}{\sqrt{2k_0'}\,(-E_{\boldsymbol{l}} - E_{\boldsymbol{l}+\boldsymbol{k}'} - k_0')} \\
& \left.\begin{aligned}
& + f^2\sum_{\boldsymbol{l}} \frac{\Phi(\boldsymbol{l}^+, \boldsymbol{l}^-, \boldsymbol{k}^{\sim})(\boldsymbol{l}^+|\gamma^4|\boldsymbol{l}^+ + \boldsymbol{k})(\boldsymbol{l}^+ + \boldsymbol{k}|\gamma^4|\boldsymbol{l}^-)}{2k_0(-E_{\boldsymbol{l}} - E_{\boldsymbol{l}})(-E_{\boldsymbol{l}} - E_{\boldsymbol{l}+\boldsymbol{k}} + k_0)} \\
& + f^2\sum_{\boldsymbol{l}} \frac{\Phi(\boldsymbol{l}^+ + \boldsymbol{k}, \boldsymbol{l}^- + \boldsymbol{k}, \boldsymbol{k}^{\sim})(\boldsymbol{l}^-|\gamma^4|\boldsymbol{l}^- + \boldsymbol{k})(\boldsymbol{l}^+ + \boldsymbol{k}|\gamma^4|\boldsymbol{l}^-)}{2k_0(-E_{\boldsymbol{l}+\boldsymbol{k}} - E_{\boldsymbol{l}+\boldsymbol{k}})(-E_{\boldsymbol{l}} - E_{\boldsymbol{l}+\boldsymbol{k}} + k_0)} \\
& + f^2\sum_{\boldsymbol{l}} \sum_{\substack{\boldsymbol{k}' \\ (=\boldsymbol{k}:\,\sqrt{2}\text{ times})}} \frac{\Phi(\boldsymbol{l}^+ + \boldsymbol{k}', \boldsymbol{l}^- + \boldsymbol{k}', \boldsymbol{k}^{\sim})(\boldsymbol{l}^+ + \boldsymbol{k}'|\gamma^4|\boldsymbol{l}^+)(\boldsymbol{l}^+|\gamma^4|\boldsymbol{l}^- + \boldsymbol{k}')}{2k_0'(-E_{\boldsymbol{l}+\boldsymbol{k}'} - E_{\boldsymbol{l}+\boldsymbol{k}'})(-E_{\boldsymbol{l}} - E_{\boldsymbol{l}+\boldsymbol{k}'} - k^{0\prime})} \\
& + f^2\sum_{\boldsymbol{l}} \sum_{\substack{\boldsymbol{k}' \\ (=\boldsymbol{k}:\,\sqrt{2}\text{ times})}} \frac{\Phi(\boldsymbol{l}^+, \boldsymbol{l}^-, \boldsymbol{k}^{\sim})(\boldsymbol{l}^- + \boldsymbol{k}'|\gamma^4|\boldsymbol{l}^-)(\boldsymbol{l}^+|\gamma^4|\boldsymbol{l}^- + \boldsymbol{k}')}{2k_0'(-E_{\boldsymbol{l}} - E_{\boldsymbol{l}})(-E_{\boldsymbol{l}} - E_{\boldsymbol{l}+\boldsymbol{k}'} - k_0')}
\end{aligned}\right\} \quad \text{(a)} \\
& + f^2\sum_{\boldsymbol{l}} \sum_{\substack{\boldsymbol{k}' \\ (=\boldsymbol{k}:\,\sqrt{2}\text{ times}) \\ (=-\boldsymbol{k}:\,\sqrt{2}\text{ times})}} \frac{\Phi(0^+, 0^-, -\boldsymbol{k}^{\sim\prime}, \boldsymbol{k}^{\sim\prime}, \boldsymbol{k}^{\sim})(\boldsymbol{l}^- + \boldsymbol{k}'|\gamma^4|\boldsymbol{l}^+)(\boldsymbol{l}^+|\gamma^4|\boldsymbol{l}^- + \boldsymbol{k}')}{2k_0'(-k_0' - k_0')(-E_{\boldsymbol{l}} - E_{\boldsymbol{l}+\boldsymbol{k}'} - k_0')} \quad \text{(b)} \\
& - \frac{1}{2} f^2 \sum_{\boldsymbol{l}} \left[\frac{(\boldsymbol{l}^-|\gamma^4|\boldsymbol{l}^+ + \boldsymbol{k})(\boldsymbol{l}^+ + \boldsymbol{k}|\gamma^4|\boldsymbol{l}^-)}{2k_0(-E_{\boldsymbol{l}} - E_{\boldsymbol{l}+\boldsymbol{k}} + k_0)} \right. \\
& \qquad \left. + \sum_{(\boldsymbol{k}'=\boldsymbol{k}:\,2\text{ times})} \frac{(\boldsymbol{l}^- + \boldsymbol{k}'|\gamma^4|\boldsymbol{l}^+)(\boldsymbol{l}^+|\gamma^4|\boldsymbol{l}^- + \boldsymbol{k}')}{2k_0'(-E_{\boldsymbol{l}} - E_{\boldsymbol{l}+\boldsymbol{k}'} - k_0')} \right] \Phi(0^+, 0^-, \boldsymbol{k}^{\sim}) \quad \text{(c)} \qquad (3\cdot 1)
\end{aligned}
$$

where (a) are states with one-electron pair, one meson and (b) is state with three mesons and (c) is the renormalization term of probability amplitude; only these contribute is easily be seen from (2·1).

Then $\delta < \int T_{\mu\nu} dV >$ becomes, after subtracting the vacuum value:

$$
\begin{aligned}
\delta < \int T_{\mu\nu} dV > = {} & -2f^2 \sum_{\boldsymbol{l}} \left[\frac{(\boldsymbol{l}^-|\gamma^4|\boldsymbol{l}^+ + \boldsymbol{k})(\boldsymbol{l}^+ + \boldsymbol{k}|\gamma^4|\boldsymbol{l}^-)}{2k_0(-E_{\boldsymbol{l}} - E_{\boldsymbol{l}+\boldsymbol{k}} + k_0)} \right. \\
& \qquad \left. + \frac{(\boldsymbol{l}^- + \boldsymbol{k}|\gamma^4|\boldsymbol{l}^+)(\boldsymbol{l}^+|\gamma^4|\boldsymbol{l}^- + \boldsymbol{k})}{2k_0(-E_{\boldsymbol{l}} - E_{\boldsymbol{l}+\boldsymbol{k}} - k_0)} \right] \delta_{\mu 4} \delta_{\nu 4} \\
& + f^2 \sum_{\boldsymbol{l}} \frac{(\boldsymbol{l}^-|\gamma^4|\boldsymbol{l}^+ + \boldsymbol{k})(\boldsymbol{l}^+ + \boldsymbol{k}|\gamma^4|\boldsymbol{l}^-)}{2k_0(-E_{\boldsymbol{l}} - E_{\boldsymbol{l}+\boldsymbol{k}} + k_0)^2} \left[-\frac{1}{k_0} \begin{Bmatrix} \boldsymbol{k} \\ ik_0 \end{Bmatrix}_{\mu} \begin{Bmatrix} \boldsymbol{k} \\ ik_0 \end{Bmatrix}_{\nu} \right.
\end{aligned}
$$

$$+(\boldsymbol{l}^-|\begin{Bmatrix}\alpha\\ i\end{Bmatrix}_\mu\begin{Bmatrix}-\boldsymbol{l}\\ iE_{\boldsymbol{l}}\end{Bmatrix}_\nu|\boldsymbol{l}^-)+(\boldsymbol{l}^++\boldsymbol{k}|\begin{Bmatrix}\alpha\\ i\end{Bmatrix}_\mu\begin{Bmatrix}\boldsymbol{l}+\boldsymbol{k}\\ iE_{\boldsymbol{l}+\boldsymbol{k}}\end{Bmatrix}_\nu|\boldsymbol{l}^++\boldsymbol{k})\Big]$$

$$+f^2\sum_{\boldsymbol{l}}\frac{(\boldsymbol{l}^-+\boldsymbol{k}|\gamma^4|\boldsymbol{l}^+)(\boldsymbol{l}^+|\gamma^4|\boldsymbol{l}^-+\boldsymbol{k})}{2k_0(-E_{\boldsymbol{l}}-E_{\boldsymbol{l}+\boldsymbol{k}}-k_0)^2}\Big[\frac{1}{k_0}\begin{Bmatrix}\boldsymbol{k}\\ ik_0\end{Bmatrix}_\mu\begin{Bmatrix}\boldsymbol{k}\\ ik_0\end{Bmatrix}_\nu$$

$$+(\boldsymbol{l}^+|\begin{Bmatrix}\alpha\\ i\end{Bmatrix}_\mu\begin{Bmatrix}\boldsymbol{l}\\ iE_{\boldsymbol{l}}\end{Bmatrix}_\nu|\boldsymbol{l}^+)+(\boldsymbol{l}^-+\boldsymbol{k}|\begin{Bmatrix}\alpha\\ i\end{Bmatrix}_\mu\begin{Bmatrix}-\boldsymbol{l}-\boldsymbol{k}\\ iE_{\boldsymbol{l}+\boldsymbol{k}}\end{Bmatrix}_\nu|\boldsymbol{l}^-+\boldsymbol{k})\Big]$$

$$-f^2\sum_{\boldsymbol{l}}\frac{(\boldsymbol{l}^++\boldsymbol{k}|\gamma^4|\boldsymbol{l}^-)(\boldsymbol{l}^-|\begin{Bmatrix}\alpha\\ i\end{Bmatrix}_\mu\begin{Bmatrix}\boldsymbol{l}\\ 0\end{Bmatrix}_\nu|\boldsymbol{l}^+)(\boldsymbol{l}^+|\gamma^4|\boldsymbol{l}^++\boldsymbol{k})}{4k_0E_{\boldsymbol{l}}(-E_{\boldsymbol{l}}-E_{\boldsymbol{l}+\boldsymbol{k}}+k_0)}+\ll\text{transposed}\gg$$

$$+f^2\sum_{\boldsymbol{l}}\frac{(\boldsymbol{l}^-|\gamma^4|\boldsymbol{l}^-+\boldsymbol{k})(\boldsymbol{l}^-+\boldsymbol{k}|\begin{Bmatrix}\alpha\\ i\end{Bmatrix}_\mu\begin{Bmatrix}\boldsymbol{l}+\boldsymbol{k}\\ 0\end{Bmatrix}_\nu|\boldsymbol{l}^++\boldsymbol{k})(\boldsymbol{l}^-+\boldsymbol{k}|\gamma^4|\boldsymbol{l}^-)}{4k_0E_{\boldsymbol{l}+\boldsymbol{k}}(-E_{\boldsymbol{l}}-E_{\boldsymbol{l}+\boldsymbol{k}}+k_0)}+\ll\text{transposed}\gg$$

$$-f^2\sum_{\boldsymbol{l}}\frac{(\boldsymbol{l}^+|\gamma^4|\boldsymbol{l}^-+\boldsymbol{k})(\boldsymbol{l}^-+\boldsymbol{k}|\begin{Bmatrix}\alpha\\ i\end{Bmatrix}_\mu\begin{Bmatrix}\boldsymbol{l}+\boldsymbol{k}\\ 0\end{Bmatrix}_\nu|\boldsymbol{l}^++\boldsymbol{k})(\boldsymbol{l}^++\boldsymbol{k}|\gamma^4|\boldsymbol{l}^+)}{4k_0E_{\boldsymbol{l}+\boldsymbol{k}}(-E_{\boldsymbol{l}}-E_{\boldsymbol{l}+\boldsymbol{k}}-k_0)}+\ll\text{transposed}\gg$$

$$+f^2\sum_{\boldsymbol{l}}\frac{(\boldsymbol{l}^-+\boldsymbol{k}|\gamma^4|\boldsymbol{l}^-)(\boldsymbol{l}^-|\begin{Bmatrix}\alpha\\ i\end{Bmatrix}_\mu\begin{Bmatrix}\boldsymbol{l}\\ 0\end{Bmatrix}_\nu|\boldsymbol{l}^+)(\boldsymbol{l}^+|\gamma^4|\boldsymbol{l}^-+\boldsymbol{k})}{4k_0E_{\boldsymbol{l}}(-E_{\boldsymbol{l}}-E_{\boldsymbol{l}+\boldsymbol{k}}-k_0)}+\ll\text{transposed}\gg$$

$$+f^2\sum_{\boldsymbol{l}}\Big[\frac{(\boldsymbol{l}^-|\gamma^4|\boldsymbol{l}^++\boldsymbol{k})(\boldsymbol{l}^++\boldsymbol{k}|\gamma^4|\boldsymbol{l}^-)}{2k_0(-E_{\boldsymbol{l}}-E_{\boldsymbol{l}+\boldsymbol{k}}+k_0)}+\frac{(\boldsymbol{l}^-+\boldsymbol{k}|\gamma^4|\boldsymbol{l}^+)(\boldsymbol{l}^+|\gamma^4|\boldsymbol{l}^-+\boldsymbol{k})}{2k_0(-E_{\boldsymbol{l}}-E_{\boldsymbol{l}+\boldsymbol{k}}-k_0)}\Big]\cdot$$

$$\frac{1}{2k_0^2}\Big[\begin{Bmatrix}\boldsymbol{k}\\ ik_0\end{Bmatrix}_\mu\begin{Bmatrix}-\boldsymbol{k}\\ ik_0\end{Bmatrix}_\nu+\begin{Bmatrix}-\boldsymbol{k}\\ ik_0\end{Bmatrix}_\mu\begin{Bmatrix}\boldsymbol{k}\\ ik_0\end{Bmatrix}_\nu+2\delta_{\mu\nu}k_0^2\Big] \qquad (3\cdot2)$$

As before, the vanishment of the self-momentum is obvious. The self-energy and the self-stress for resting meson ($\boldsymbol{k}=0$) become as follows:

$$\delta<\int\mathring{T}_{44}\,d\mathring{V}>=f^2\sum_{\boldsymbol{l}}\frac{4\boldsymbol{l}^2}{\mu E(4E^2-\mu^2)} \qquad (3\cdot3)$$

$$\delta<\int\mathring{T}_{11}\,d\mathring{V}>=f^2\sum_{\boldsymbol{l}}\frac{4\boldsymbol{l}^2(4E^2+\mu^2)}{\mu E^3(4E^2-\mu^2)^2}\,\boldsymbol{l}_1\cdot\boldsymbol{l}_1$$

$$-f^2\sum_{\boldsymbol{l}}\frac{8m^2}{\mu E^3(4E^2-\mu^2)}\,\boldsymbol{l}_1\cdot\boldsymbol{l}_1-f^2\sum\frac{4\boldsymbol{l}^2}{\mu E(4E^2-\mu^2)} \qquad (3\cdot4)$$

Self-energy and self-stress contain quadratic and logarithmic divergences. Of course, the quadratic divergences of (4·3) and (4·4) diminishes to logarithmic one by the redefinition of $\Delta^{(1)}$-function as Schwinger or else.[7]

From the gauge consideration of the self-energy of photon, Pauli and Villars[9] proposed to regularize the expression of photon self-energy by the regulator with conditions

$$\sum_{i=0}^{\infty} C_i=0, \quad \sum_{i=0}^{\infty} C_i m_i^2=0, \quad \sum_{i=0}^{\infty} C_i \log m_i^2=0, \quad \sum_{i=0}^{\infty} C_i\, m_i^2 \log m_i^2=0, \quad C_0=1, \quad m_0=m \tag{3·5}$$

So that, we should also apply this to the expression (3·3), (3·4) in consistent way to photon case.

Then, (3·3) and (3·4) becomes as follows:

$$\delta < \int \mathring{T}_{44}\, d\mathring{V} > = \mu \times (\text{finite}) \tag{3·6}$$

$$\delta < \int \mathring{T}_{11}\, d\mathring{V} > = 0 \tag{3·7}$$

The result (3·7) is obtained by applying the regulator (3·5) to the expression (3·4), and since the ex pression (3·4) is obtained as an expectation value of total energy-momentum tensor, so that the result (3·7) is not the consequence of purely formalistic reguralization procedure, but consequence of realistic mixture with negative energy Fermion field with mass infinite.*** Both expression give finite value but do not vanish whe $\mu \neq 0$. Only for $\mu=0$, the regularized expression vanihses, and thus the mass zero meson can be described as stable relativistic particle.

4. Self-Stress of an Electron due to Pseudo-Scalar-Meson Field

The calculations are thoroughly similar to 2, only changes come for $\gamma^4 \to \gamma^4\gamma^5$, $f \to if$. We only give our results:

$$\delta < \int \mathring{T}_{44}\, d\mathring{V} > = -f^2 \sum_{\boldsymbol{k}} \frac{1}{4k_0 E}\left[\frac{E-m}{m-E-k_0} - \frac{E+m}{-m-E-k_0}\right]$$

$$\begin{aligned}
\delta < \int \mathring{T}_{11}\, d\mathring{V} > = {} & f^2 \sum_{\boldsymbol{k}} \frac{1}{3}\left[\frac{1}{4k_0 E^2} + \frac{1}{4k_0^2 E}\right]\left[\frac{E-m}{(m-E-k_0)^2} - \frac{E+m}{(-m-E-k_0)^2}\right]\cdot k^2 \\
& + f^2 \sum_{\boldsymbol{k}} \frac{1}{3}\cdot\frac{m}{4k_0 E^3}\left[\frac{1}{m-E-k_0} + \frac{1}{-m-E-k_0}\right]\cdot k^2 \\
& - f^2 \sum_{\boldsymbol{k}} \frac{1}{3}\cdot\frac{1}{4k_0^3 E}\left[\frac{E-m}{m-E-k_0} - \frac{E+m}{-m-E-k_0}\right]\cdot k^2 \\
& + f^2 \sum_{\boldsymbol{k}} \frac{1}{4k_0 E}\left[\frac{E-m}{m-E-k_0} - \frac{E+m}{-m-E-k_0}\right]
\end{aligned} \tag{4·1}$$

which becomes for $\mu=0$:

$$\delta < \int \mathring{T}_{11}\, d\mathring{V} >_{\mu=0} = \frac{(f^2/4\pi)}{12\pi} \tag{4·2}$$

and the same value is obtained for $\mu \neq 0$ also.**

5. Self-Energy and Self-Stress of Pseudo-Scalar-Meson due to Electron-Positron pairs

By the mentioned change in preceding paragraph, we ontain:

$$\delta < \int \mathring{T}_{44} \, d\mathring{V} > = f^2 \sum_{\boldsymbol{l}} \frac{4E}{\mu(4E^2-\mu^2)} \tag{5·1}$$

$$\delta < \int \mathring{T}_{41} \, d\mathring{V} > = 0 \tag{5·2}$$

$$\begin{aligned} \delta < \int \mathring{T}_{11} \, d\mathring{V} > &= f^2 \sum_{\boldsymbol{l}} \frac{4(4E^2+\mu^2)}{\mu E(4E-\mu^2)^2} \boldsymbol{l}_1 \cdot \boldsymbol{l}_1 \\ &\quad - f^2 \sum_{\boldsymbol{l}} \frac{4E}{\mu(4E^2-\mu^2)} \end{aligned} \tag{5·3}$$

$$\delta < \int \dot{T}_{44} \, d\dot{V} > = \mu \times (\text{finite}) \tag{5·4}$$

$$\delta < \int \dot{T}_{11} \, d\dot{V} > = 0 \tag{5·5}$$

The result (5·5) is obtained in the same way as (3·7). The mentioned change for the meaning of regularization is necessary.*** This also vanishes for $\mu=0$, otherweise finite.

In this case, also, the mass zero meson can only be described as stable relativistic particle.

6. Self-Stress of an Electron due to Vector Meson field. (Radiation field)

Though this calculation was firstly done by Pais[1], we summarize our calculations here for completeness.

The calculations are similar to 2, by change of $\gamma^4 \rightarrow \gamma^4\gamma^\alpha$, $f \rightarrow ie$, and summation over α.

The part $\delta < \int T_{44} \, dV >$ is minus sign of familar expression of self-energy of an electron due to radiation field[9], and self-momentum vanishes also: The expression for resting electron becomes as follows:

$$\delta < \int \mathring{T}_{44} \, d\mathring{V} > = -e^2 \sum_{\boldsymbol{k}} \frac{1}{2k_0 E} \left[\frac{E-2m}{m-E-k_0} - \frac{E+2m}{-m-E-k_0} \right]$$

$$\begin{aligned} \delta < \int \mathring{T}_{11} \, d\mathring{V} > &= e^2 \sum \frac{1}{3} \cdot \left[\frac{1}{2k_0 E^2} + \frac{1}{2k_0^2 E} \right] \left[\frac{E-2m}{(m-E-k_0)^2} - \frac{E+2m}{(-m-E-k_0)^2} \right] \cdot k^2 \\ &\quad + e^2 \sum \frac{1}{3} \cdot \frac{m}{k_0 E^3} \left[\frac{1}{m-E-k_0} + \frac{1}{-m-E-k_0} \right] \cdot k^2 \end{aligned}$$

$$-e^2\sum\frac{1}{3}\cdot\frac{1}{2k_0^3E}\left[\frac{E-2m}{m-E-k_0}-\frac{E+2m}{-m-E-k_0}\right]\cdot k^2$$

$$+e^2\sum\frac{1}{2k_0E}\left[\frac{E-2m}{m-E-k_0}-\frac{E+2m}{-m-E-k_0}\right] \tag{6·1}$$

$$\delta<\int\mathring{T}_{11}\,d\mathring{V}>=\frac{(e^2/4\pi)}{2\pi}m=\frac{\alpha}{2\pi}m \quad \text{(for any } \mu)^{**} \tag{6·2}$$

and just corresponds to Pais's result.1) By the addition of this and (2·5) with condition $f^2=2e^2$, one obtains vanishing self-stress in f-field theory.

7. General Relation between Self-Energy and Self-Stress*

Recently, Pais and Epstein10) found very interesting relation between the self-stress and self-energy of an electron interacting with radiation field, by using the fact that the trace of the symmetrical energy momentum tensor of radiation field is zero;

$$<\bar{T}_{11}>=-\frac{1}{3}\left\{<\bar{T}_{44}>-m\frac{\partial}{\partial m}<\bar{T}_{44}>\right\} \tag{7·1}$$

But, regardless with these special circumstances, we found the following relation between self-stress and self-energy of Fermion due to Boson and Boson due to Fermion;

$$<\bar{T}_{11}>=-\frac{1}{3}\left\{<\bar{T}_{44}>-m\frac{\partial}{\partial m}<\bar{T}_{44}>-\mu\frac{\partial}{\partial\mu}<\bar{T}_{44}>\right\} \tag{7·2}$$

where m is the Fermion mass and μ is Boson mass. This relation includes Pais-Epstein relation (7·1) as the special case of Boson mass zero. The self-stress of Fermion due to scalar-, pseudo-scalar- and vector-meson fields can be expressed in terms of self-energy by (7·2) can be easily proved by using the results of direct calculations, (2·4), (4·1) and (6·1).

Moreover, the self-stress of Boson due to Fermi particle also satisfies (7·2) as is easily proved from (3·4) and (5·3).

Thus, the relation (7·2) holds for the interacting system of mass m Fermion and mass μ Boson, in general.

The reasonable physical interpretation of the relation (7·2) is not yet made, and will be discussed later on.

8. A consideration on the origin of self-stress

Why the self-stress appearr? This question cannot be answered easily. But, by theoretical consideration we can naturally conclude that its appearance originate to the non commutability of the operator of self-stress with that of interactions. And we can show that it comes from the surface-integral in $k^{(4)}$-space around the singularity and far-away boundary. To show this, we take Feynman notation, and as example, consider the self-stress of electron due to scalar, pseudoscalar and vector meson fields.

Reproduced in Feynman notation, one obtains for the trace of the energy-momentum tensor of one-electron state;

$$\sum_{\mu=1}^{4}<\bar{T}_{\mu\mu}>=<\bar{T}_{44}>-\sum^{3}<\bar{T}_{ii}>$$

$$=\frac{g^2}{\pi i}\int\gamma^{\cdot\cdot}(\bar{p}-\bar{k}-m)^{-1}(\bar{p}-\bar{k})(\bar{p}-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}\frac{d^{(4)}k}{k^2-\mu^2}$$

$$-\frac{g^2}{\pi i}\int\gamma^{\cdot\cdot}(\bar{p}-\bar{k}-m)^{-1}\frac{\gamma^4m^2}{E_p}(\bar{p}--\bar{k}-m)^{-1}\gamma^{\cdot\cdot}\frac{d^{(4)}k}{k^2-\mu^2}$$

$$+\frac{g^2}{\pi i}\int(4\mu^2-2k_0k_0)\gamma^{\cdot\cdot}(\bar{p}-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}\frac{d^{(4)}k}{(k^2-\mu^2)^2}$$

$$+\frac{g^2}{\pi i}\int\gamma^{\cdot\cdot}(\bar{p}-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}\frac{d^{(4)}k}{k^2-\mu^2} \qquad (8\cdot1)$$

here, $<\bar{T}_{44}>$ is just that given by perturbation calculation, $<\bar{T}_{11}>$, $<\bar{T}_{22}>$, $<\bar{T}_{33}>$ have, opposite sign to the latter. And, $g, \gamma^{\cdot\cdot}$ are characteristic for the type of interaction: g, $i\gamma^{\alpha}$ for vectormeson field, f, 1 for scalar meson field, f, $i\gamma^5$ for pseudoscalar meson field. The mass of meson is μ. The last term of the expression $(8\cdot1)$ is just $<\bar{T}_{44}>$.

The general relations between self-energy and self-stress $(7\cdot2)$ can be proved easily in this form also.

Now, one can see that the second expression is just canceled by a part of the first expression, and sum of the first and second becomes:

$$-\frac{g^2}{\pi i}\int\gamma^{\cdot\cdot}(\bar{p}-\bar{k}-m)^{-1}\bar{k}(\bar{p}-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}\frac{d^{(4)}k}{k^2-\mu^2}$$

$$=-\frac{g^2}{\pi i}\int\frac{\partial}{\partial k_\lambda}\{\gamma^{\cdot\cdot}(\bar{p}-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}\}\cdot\frac{k_\lambda}{k^2-\mu^2}d^{(4)}k \qquad (8\cdot2)$$

$$=-\frac{g^2}{\pi i}\int\frac{\partial}{\partial k_\lambda}\left\{\gamma^{\cdot\cdot}(\bar{p}-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}\frac{k_\lambda}{k^2-\mu^2}\right\}d^{(4)}k$$

$$+\frac{g^2}{\pi i}\int(2k^2-4\mu^2)\gamma^{\cdot\cdot}(\bar{p}-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}\frac{d^{(4)}k}{(k^2-\mu^2)^2} \qquad (8\cdot3)$$

So that $(8\cdot1)$ becomes;

$$\sum_{\mu}<\bar{T}_{\mu\mu}>=-\frac{g^2}{\pi i}\int\frac{\partial}{\partial k_\lambda}\left\{\gamma^{\cdot\cdot}(\bar{p}-\bar{k}-m)^{-1}\gamma^{\cdot\cdot}\frac{k_\lambda}{k^2-\mu^2}\right\}d^{(4)}k+<\bar{T}_{44}> \qquad (8\cdot4)$$

since the last of expression $(8\cdot1)$ is $<\bar{T}_{44}>$.

This shows that

$$\sum_{i=1}^{3} < \bar{T}_{ii} > = \frac{g^2}{\pi i} \int \frac{\partial}{\partial k_\lambda} \left\{ \gamma^{\cdot\cdot} (\bar{p} - \bar{k} - m)^{-1} \gamma^{\cdot\cdot} \frac{k_\lambda}{k^2 - \mu^2} \right\} d^{(4)} k \qquad (8\cdot5)$$

and is essentially integral over the $k^{(4)}$-space surface around singularity and boundary.

All other cases of self-stress of Boson field due to Fermion pair-creation-annihilation can also be expressed in analogous form as $(8\cdot5)$ by the surface integral.**

If one wants to eliminate the self-stress, one should make some assumption on the singular surface and on the boundary of $k^{(4)}$-space. But this assumption will be concerned with the dimnish of the self-energy to not contain logarithmic divergence and only contain terms mass times constant,[10] which is evident from the general relation $(7\cdot2)$.

These assumptions are admissible or not is now being examined.

9. Conclusions

From the above examples, we can firstly say that the realistic subtraction procedure of C-meson or f-field is not yet fully consistent as to the self-stress of C-meson itself and due to the peculiar behavior of self-energy of C-meson, though both of these can be brought into finite value by applying the regulator which was used to secure the gauge covariance in the photon self-energy.*** Only when the mass of meson is zero, it has no self-stress and -energy (by applying regulator), in nowadays theory, and can be ascribed as a stable relativistic particle. The theory with vanishing mass C-meson is surely closed, since from $(2\cdot5)$ $(6\cdot2)$ and $(3\cdot6)$ $(3\cdot7)$ the self-stress of cencerning particles are all zero, photons and C-meson has no self-energy, and only electron has self-energy of an amount $-\frac{\alpha}{4\pi} m^{11}$, which can be safely renormalized into the theoretical mass, not affecting the self-stress as has shown in the introduction. But this theory is also unsatisfactory since it modifies well-established accordance between theoretical and experimental data considerably.

From these considerations, we may say that in the present frame of theory, the field with spin 0 or 1 is not essentially relativistic particle if it has any mass (intrinsic or else), only spin ½ particle can be ascribed as stable relativistic particle with mass.

As to the general relation between self-stress and self-energy proved in paragraph 7, we cannot yet interpret it reasonably. The physical considerations onto this relation and related problems will be discussed later.

In conclusion, I say much thanks to Prof. M. Kobayasi for his kind interest in this work.

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* This paragraph was attached after the Annual Meeting of Japan Physical Society, Kyoto, Oct. (1949), where I heard of the paper of Pais and Epstein from Mr. G. Takeda.

** See. Part II. Prog. Theor. Phys. **5**. in press.

*** See. Part II. § 4 (C).

Letters to the Editor

Linearization of Minkowski Space and Five-dimensional Space

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October 17, 1949

From the far-reaching symmetry between space-time and momentum-energy in quantum theory⁽¹⁾, it is desirable to develope the whole theory in the linearized space-time corresponding to the linearization of momentum-energy as done by Dirac.

In this letter, Maxwell equation and Proca's equation are difined in linear space-time world, just as Cauchy-Riemann's differential equation is satisfied by analytic function of complex variable $x+iy$ or linearized two dimensional Euclid space. Maxwell equation can be written in the form of the relations of pseudo-quaternion or linearized pseudo-Euclid space or Minkowski space just as Hamilton's quaternion is the linearization of four-dimensional Euclid space. The vector analysis used in classical mechanics and electrodynamics in Minkowski space, from this point of view, did not properly take into consideration the distinction between time and space-components of four-vector and their resemblance.

To a point $P(x_0, x_1, x_2. x_3)$ in Minkowski space corresponds a pseudo-quaternion $R=\sum_{\mu=0}^{3} a_\mu x_\mu$, and R^* is the conjugate defined by $R^*=\sum_{\mu=0}^{3} a_\mu^* x_\mu$, where $a_0=1$, a_1, a_2 and a_3 satisfy the same commutation relations as Dirac's matrix and $a_k^* = -a_k (k=1, 2, 3)$. Then $R^*R=x_0{}^2-x_1{}^2-x_2{}^2-x_3{}^2$ holds. Maxwell equation reduces to $D^*A^*=-F^*, D(-F^*)=S^*$ and Lorentz condition $-E_0=\sum_{\mu=0}^{3}\partial A_\mu/\partial x_\mu=0$ where

$$D=\sum_{\mu=0}^{3} a_\mu(\partial/\partial x_\mu),\ A=\sum_{\mu=0}^{3} a_\mu A_\mu,$$

$$F=E_0+\sum_{k=1}^{3} a_k(E_k+iH_k),$$

$$S=4\pi\sum_{\mu=0}^{3} a_\mu I_\mu \text{ and } i=\sqrt{-1}.$$

Also the above equations can be regarded as matrix equation, when

$$a_1=\left\{\begin{matrix}0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -i\\ 0 & 0 & i & 0\end{matrix}\right\},\quad a_2=\left\{\begin{matrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & i\\ 1 & 0 & 0 & 0\\ 0 & -i & 0 & 0\end{matrix}\right\},$$

$$a_3=\left\{\begin{matrix}0 & 0 & 0 & 1\\ 0 & 0 & -i & 0\\ 0 & i & 0 & 0\\ 1 & 0 & 0 & 0\end{matrix}\right\},\quad A=\left\{\begin{matrix}A_0\\ A_1\\ A_2\\ A_3\end{matrix}\right\},$$

$$F=\left\{\begin{matrix}E_0\\ E_1+iH_1\\ E_2+iH_2\\ E_3+iH_3\end{matrix}\right\},\quad S=\left\{\begin{matrix}4\pi I_0\\ 4\pi I_1\\ 4\pi I_2\\ 4\pi I_3\end{matrix}\right\}.$$

For more symmetrical from, consider A, F and S are complex quantities and equal to

$$A=\left\{\begin{matrix}A_0+iB_0\\ A_1+iB_1\\ A_2+iB_2\\ A_3+iB_3\end{matrix}\right\},\quad F=\left\{\begin{matrix}E_0+iH_0\\ E_1+iH_1\\ E_2+iH_2\\ E_3+iH_3\end{matrix}\right\},$$

$$S=4\pi\left\{\begin{matrix}I_0+iJ_0\\ I_1+iJ_1\\ I_2+iJ_2\\ I_3+iJ_3\end{matrix}\right\},$$

then we obtain the following equations:

$$-E_0=\partial A_0/\partial t+\mathrm{div}\boldsymbol{A},$$

$$\boldsymbol{E}=-\partial\boldsymbol{A}/\partial t-\mathrm{grad}A_0-\mathrm{curl}\boldsymbol{B},$$

$$-H_0=\partial B_0/\partial t+\mathrm{div}\boldsymbol{B},$$

$$\boldsymbol{H}=-\partial\boldsymbol{B}/\partial t-\mathrm{grad}B_0+\mathrm{curl}\boldsymbol{A},$$

$$I_0=-\partial E_0/\partial t+\mathrm{div}\boldsymbol{E},$$

$$\boldsymbol{I}=-\partial\boldsymbol{E}/\partial t+\mathrm{grad}E_0+\mathrm{curl}\boldsymbol{H},$$

$$J_0=-\partial H_0/\partial t+\mathrm{div}\boldsymbol{H},$$

$$\boldsymbol{J}=-\partial\boldsymbol{H}/\partial t+\mathrm{grad}H_0-\mathrm{curl}\boldsymbol{E},$$

where J_0 and $\boldsymbol{J}$ are magnetic charge and current density.

The difference between the theory of

function of pseudo-quaternion and that of complex variable is as follows;

(I) There exist nilfactor (that is light cone),
(II) Two pseudo-quaternions do not in general commute,
(III) Four dimensional space.

In the case of Hamilton's quaternion nilfactor does not exist and the second order differential equation becomes elliptic type. But the existence of nilfactor changes the fundamental character of the theory and the second order differential equation is hyperbolic type. This is the source of the separation of space-like and time-like region. We see this clearly by introducing new coordinate defined by

$$R=Y_{\xi}\xi+Z_{\eta}\eta$$

where

$$Z_{\xi}=x_0+r,\quad Z_{\eta}=x_0-r,\quad \xi=(1+\sum_{k=1}^{3}a_kx_k/r)/2,$$

$$\eta=(1-\sum^{3}a_kx_k/r)/2 \text{ and } r=(\sum^{3}x_k^2)^{1/2}.$$

Then the function of R splits into two parts:

$$F(R)=\xi F(Z_{\xi})+\eta F(Z_{\eta})$$

owing to the following relations, $\xi^n=\xi$, $\eta^n=\eta$, and $\xi\eta=0$. The theorem corresponding to Cauchy's integral theorem is as follows:

$$\int_{(H)}-DF(R)dh=\int_{(H)}S^*dh=-\int_{(R)}dR\cdot F(R)$$

where (R) is the three dimensional surface of four dimensional volume (H). When $S^*=0$, $F(R)$ is called left-regular and when $G(R)$ is right-regular, $\int_{(R)}G(R)dR=0$ and $\int_{(R)}G(R)dR\times F(R)=0$ hold.

The solution of the above equation. $F(R)$ is given by $F(R)=(\partial^2/\partial x_0^2-\sum_{k=1}^{3}\partial^2/\partial x_k^2)\,W(R)$, where $W(R)$ is an analytic function of R.

In five dimensional case, we define five dimensional number or linear five dimensional space $R=\sum_{\mu=0}^{3}a_\mu x_\mu+\beta x_5$ in which distance is $|R|^2=x_0^2-\sum_{k=1}^{3}x_k^2-x_5^2$.

where a_μ and β satisfy the same commutation relations as Dirac's matrix a_μ and β. Then Proca's equation reduces to

$$D^*A^*=-F^*,\quad -DF^*=S^*,\quad G^0=0$$

and

$$i\hbar\frac{\partial}{\partial x_5}=mc,$$

where $D=\sum_{\mu=0}^{3}a_\mu\,\partial/\partial x_\mu+\beta\frac{\partial}{\partial x_5}$

$$A=\sum_{\mu=0}^{3}a_\mu A_\mu+\beta A_5$$

$$F=G_0+\mathbf{F}+i\mathbf{G}+\beta U_0+\beta\mathbf{U},$$

$$S^*=M_0+\mathbf{M}+\beta\mathbf{T}+i\beta\mathbf{S}.$$

$\mathbf{F}$, $\mathbf{G}$, U_0, $\mathbf{U}$, M_0, $\mathbf{M}$, $\mathbf{T}$ and $\mathbf{S}$ are the same notation as Yukawa's paper.(2) From the above relations we can avail the theory of function of hypercomplex variable to the solution of Maxwell equation and Proca's equation.

Further development of the theory will be given in near future.

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Note on the High Energy Neutron-Proton Scattering

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We calculated the neutron proton scattering cross section at 100 MV and 300 MV incident energy. Phase shift method was employed and the exact results were obtained. To treat relativistically, Dirac's equation of two body problem with the relativistic interaction must be solved. Separation of variables was performed according Kemmer's method and the interaction potential takes the form

$$V=\sum f_i\,\omega_i\,\delta(x)$$

$$\left.\begin{array}{ll}\omega_1=\beta^N\beta^p & \omega_4=\sigma^N\sigma^p-\Gamma^N\Gamma^p\\ \omega_2=1-\alpha^N\alpha^p & \omega_5=\beta^N\beta^p\Gamma^N\Gamma^p\\ \omega_3=\beta^N\beta^p\,[\sigma^N\sigma_p+\alpha^V\alpha^p] & \Gamma=-i\alpha_x\alpha_y\alpha_2\end{array}\right\}$$

α_c^N, α_y^N, α_z^N, β^N, α_c^P, α_y, α_z, β are Dirac's spin matrix of neutron and proton, respectively. $\delta(x)$ was approximated by the usual square well with the range of 2.8×10^{-13} cm. This procedure is incomplete from the relativistic point of view, but by this calculation, we can see the effect of ρ_1 and ρ_2 in the interaction, to the high energy scattering phenomena. The depth of the square well was determined as following: if we make $\rho_1, \rho_2\to0$, $\rho_3\to1$ in non-relativistic limit, we obtain

$$V=(f_1+f_2)+(f_3+f_4)\ \sigma^N\sigma^p$$

and this potentials are to be equate with the triplet and singlet depth which have been determined as 21.213 MV and 11.904 MV respectively, from the experimental fact of deuteron.(2)(3) So there are four possibilities about the interaction constants

$$\begin{array}{ll}\text{(I)}\ f_2=f_4=0 & \text{(II)}\ f_2=f_3=0\\ \text{(III)}\ f_2=f_4=0 & \text{(IV)}\ f_2=f_4=0\end{array}$$

and ω_5 interaction being omitted for simplicity. The values of the phase shift were determined by the continuity of the radial component of the probability flow at the edge of the square well. We calculated the total cross section and $R\fallingdotseq\sigma(\pi)/\sigma(\pi/2)$ at 100 MV and 300 MV incident energies in the case of the symmetrical theory. Four types of the potential mentioned above, give the almost same results

$$\sigma=0.11\times10^{-24}\ \text{cm}^2,\quad R=10\quad \text{at 100 MV.}$$

$$\sigma=0.06\times10^{-24}\ \text{cm}^2,\quad R\fallingdotseq300\quad \text{at 300 MV.}$$

The values of R at 100 MV are largely reduced comparing with R for the usual potentials.(4) The higher orbital states were combined with the lower through 3- and 4-th components of spinor as in the case of the tensor forces and the value of R was reduced. But at 300 MV, R is too large. Recent experimental value at 280 MV(5) was reported as about 6. The total cross section is slightly larger than the experimental value. The more detailed discussion will be published shortly.

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Density Matrix and New Configuration Space

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October 25, 1949.

A program for a formulation of the density matrix(1) in a new configuration space(2)(6) will be given in the following. In this formulation the second quantization will reveal its essential merit of simplifying and unifying the current treatments in the many body problem of quantum mechanics.

It will be shown how the theory of symmetric algebra(3) serves to determine terms of multiplets of the system under consideration. We consider operators which are composed of creation (with asterisk*) and annihilation operators (without asterisk*) operating on occupation observables c(...) in the following manner:

$$c(r\rho_1 s\rho_2,\ \ldots\ldots,\ t\rho_n)=a^*_{r\rho_1}\ a_{s\rho_2}\cdots\cdots a^*_{t\rho_n}\ a_{t'\rho_n'}\cdots\cdots a_{s'\rho'_2}\ a_{r'\rho'_1}\ c(r'\rho'_1, s'\rho_2', \ldots\ldots, t'\rho'_n)$$

Where each subindex indicates intrinsic states (spin states) taking on values from l to k. Elements of the form $\sum\limits_{\rho_1<\rho_2<\ldots<\rho_n} a^*_{r\rho_1}\ a^*_{s\rho_2}\cdots\cdots a^*_{t\rho_n}$

$a_{t'\rho_n}\cdots\cdots a_{s'\rho_2}\,a_{r'\rho_1}$ generate the so-called symmetric algebra belonging to a prescribed orbital configuration and are denoted by $S\begin{pmatrix} r, & s\ldots t \\ r' & s'\ldots t' \end{pmatrix}$ or $SP^{-1}Q$, where P and Q mean permutations of orbital states $r_1, r_2, \cdots\cdots, r_n$ of the configuration in question. These operators are not considered in the configuration space as operators operating on occupation observables, but in an algebra whose idempotents are $N_r(\lambda)=\sum\limits_{\rho_1<2<\ldots<\rho_k}^{1\to k} N_{r\rho_1}\cdots\cdots N_{r\rho_\lambda} N^*_{r\rho_{\lambda+1}}\cdots\cdots N^*_{r\rho_k}$ where $N_{r\rho}=a^*_{r\rho}a_{r\rho}$ and $N^*_{r\rho}=a_{r\rho}\,a^*_{r\rho}=(1-N_{r\rho})$. This algebra is a simple algebra of degree $f_{Ns\lambda}$ marked by a certain magnetic quantum number M_S and term of multiplets λ of the intrinsic states. (spin angular momentum S, Wigner's supermultiplets (S, T, Y), etc.).

Next we consider an algebra over Hilbert space composed of wave functions $\varphi(x)^*$ and $\varphi(x)$. This algebra is called density algebra, whose elements are defined by,

$$\rho(x_1, x'_1)=\varphi(1')^*\varphi(1), \quad \rho(x, x')=\frac{1}{n!}\varphi(1')^*\ldots \varphi(n')^*\varphi(n)\ldots\varphi(1)$$

where $\varphi(i')^*=\sum\limits_{r,\rho}a^*_{r\rho}\,\Psi_{r\rho}(i')$, $\varphi(i)=\sum\limits_{r,\rho}a_{r\rho}\,\Psi_{r\rho}(i)$ and Ψ_s are elements of Hilbert space.

We shall deal with Fermi particles with 2ν intrinsic degrees of freedom (or with $\nu-1/2$ spin). Owing to the orthogonality of the intrinsic states, the density corresponding to a certain set of orbital states is,

$$U^{(r)}\rho(x_n, x_n')\,U^{(r)},$$

where $U(r)$ is Ostertag's restriction operator corresponding to the orbital configuration (r) which is defined by $U^{(r)}=N_1(r_1)N_2(r_2)\cdots\cdots N_n(r_n)$.

$$\sum_{(r)}\text{trace } U^{(r)}\rho(x_n, x_n')\,U^{(r)}=\frac{1}{n!}\sum_{(r)}\sum_{\substack{P,Q\\ \text{mod } \pi_{(r)}}}\text{trace } Z^{(r)}{}_{PQ}\ Q\psi(x_n)\overline{P\psi(x'_n)}$$

where $\pi_{(r)}$ is the subgroup determined by the configuration (r).

Boltzmann factors are introduced by attaching the factors to a_r and the commutation relation becomes $[a_r\ a_s^*]_+=\exp(-\beta E_r)\,\delta_{rs}$. Z_P is an element of the centrum of the symmetric algebra corresponding to the orbital configuration (r), and the trace is considered in the simple algebra of degree $f_{Ns\nu}$.

Making use of the relation between an arbitrary centrum ($Z^{(r)}$) and the unique centrum corresponding to the orbital configuration giving the normal representation, we can write down the density matrix in the following form

$$\sum_{(r)}\text{trace } U^{(r)}\rho(x_n, x_n')U^{(r)}=\frac{1}{n!}[\chi_\nu(E)\,[\varphi(1)\varphi(1')^*]_+[\varphi(2)\varphi(2')^*]_+\cdots[\varphi(n)\varphi(n')^*]_+ + \cdots+\chi_\nu(P)\,\{[\varphi(1)\varphi(i_1')^*]_+[\varphi(2)\varphi(i_2')]_+\cdots[\varphi(n)\varphi(i_n')^*]_+ \text{ summed over a class of } P=\begin{bmatrix}1, 2, \cdots n\\ i_1' i_2' \cdots i'_n\end{bmatrix}\}+\cdots].$$

where $\chi_\nu(P)$ is a character of the symmetric group of n particles and $[A, B]=AB+BA$.

And the reduced density algebra is definen by

$$\rho^{(m)}(1\ldots m, 1'\ldots m')=\int\rho(1\ldots m\, m+1\ldots n, 1'\ldots m'\, m+1\ldots n)\, d(m+1)\ldots dn$$

and the density algebra for the canonical ensemble is given by

$$\rho_c=\exp(-\xi n-\beta nE)$$

where $n=\sum N_{r\rho}$ and $n\cdot E=\sum N_{r\rho}\ E_r$.

We can introduce the degeneration parameter and obtain the grand partition function and the grand density matrix for the Fermi particles with k intrinsic degrees of freedom

$$D(\xi\beta)=\text{trace } \rho_c$$

$$D(x_1 x'\xi\beta)=\text{trace } \rho\ (x_m, x_m)\rho/D(\xi\beta)$$

Noticing that $N(p)\ n\ N(p)=pN(p)$, we get

$$D(\xi\beta)=\prod_r^{\infty}\frac{1-e^{-(k+1)(\xi+\beta E_r)}}{1-e^{-\xi-\beta E_r}}$$

In the above formulation the idempotents $N_{1p_{11}}\cdots N_{1p_{1r_1}}\cdots n_{p_{n1}}\cdots n_{r_n} N_{1p_{1r_1+1}}\cdots N^*_{1p_{1k}}\cdots N^*_{np_{nr_n}}\cdots N^*_{np_{nk}}$ take place of the occupation observables.

These idempotents may be considered as spanning a new configuration space was at first given by P. Jordan.⁽⁶⁾ A similar idea to ours has been proposed by Y. Nambu⁽⁷⁾, but we cannot yet tread in the detailed discussion.

Such a compact description given by the density algebra enables us to obtain a clear perspective of the whole theory of the density matrix.

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Meson Production by X-ray

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Recent experiments carried out at Berkeley are said to have shown that the yield of negative π-meson by X-ray is about 1.4 times as large as that of positive ones, and also that the angular distribution is exactly proportional to $sin^2\ \theta$. ¹⁾

In order to compare these results with theoretical predictions, we have set out to calculate in the completely relativistic perturbation method of Feynman-Dyson.²⁾ To begin with, we have taken the pseudoscalar meson with pseudoscalar coupling and have evaluated the cross-section of this photo-mesonic effect to the order of e^2f^2. When, for example, the incident photon energy is equal to $2\varkappa C^2$ in the laboratory system ($\varkappa=\pi$-meson mass$=$286 electron mass), the total cross section for the production of positive and negative mesons

$$\sigma_-=2.0(e^2/4\pi\hbar c)\cdot(f^2/4\pi\hbar c)r_0^2,$$
$$\sigma_+=1.4(e^2/4\pi\hbar c)\cdot(f^2/4\pi\hbar c)\cdot r_0^2,$$

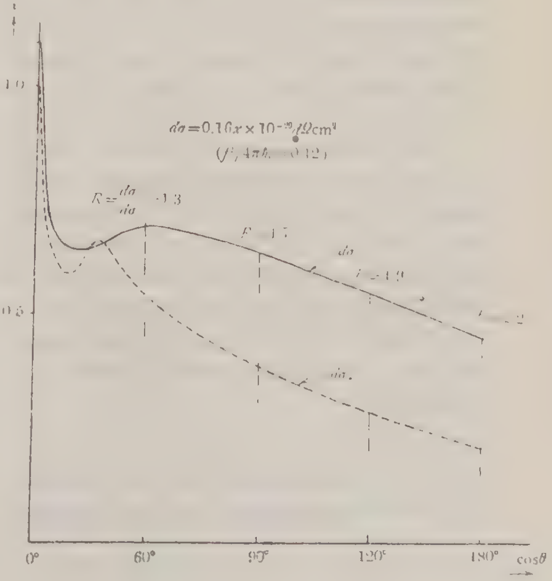
respectively, with

$$r_0^2=(\hbar/\varkappa c)^2\times 10^{-2}=7.22\cdot 10^{-27}\mathrm{cm}^2,$$

their ratio being

$$\sigma_-/_+\sigma\approx 1.4,$$

in fair agreement with the experiments. It is the relativistic effect (e.g. the recoil of the nucleon) that gives rise to this charge asymmetry. If we put $f^2/4\pi\hbar c\sim 0.1$, say, the total cross section becomes of the order 10^{-29} cm², which, too, seems not unreasonable. On the other hand, the angular distribution, which is shown in the figure, has turned out to be irreconcilable with the experimental results; and we should like to add some remarks on this point.



The effect under consideration is composed of two distinct processes due to i) the electric current of the meson, and ii) the electric current of recoil nucleon, and the contribution of the former to the cross-section is almost proportional to $sin^2\theta$ near the threshold, while it is not the case with the latter. Now in the pseudoscalar meson theory the mesons are very strongly bound to nucleon and the meson electric current is rather weak, so that its effect remains comparable with that of nucleon current, whence the above discrepancy. This depends of course on our special choice of meson type. As is already known,[3] the scalar meson theory affords the $sin^2\theta$ distribution near threshold, because in this case the term owing to the meson electric current is about 15 times as large as that of nucleon current. Thus it will be worth while to proceed to the e^2f^4-approximation, where the pseudoscalar meson current may be expected to become larger. It is to be noticed that this situation closely resembles the difficulties of nucleon magnetic moment, that has been analzed by Case.[4] Another possibility to get out of the disagreement might be to replace the free nucleon model used in our calculation by a properly bound nucleon; that would suppress the nucleon electric current. The distortion of the meson wave function at the nucleus may have some appreciale effect, too.

More comprehensive calulations, taking into account the above considerations are intended. Fuller accounts will be published later in this journal. Concluding, we express our sincere thanks to Prof. Tomonaga for his kind informations from abroad and also to Prof. Husimi, Dr. Miyazima, Mr. Hayakawa, Mr. Nambu and others for their interst and discussions.

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4) K. M. Case, Phys. Rev. **76** (1949), 1.

A Note on the Mixed Shower

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The soft component in the mixed shower is partly explained by the charge acceleration,[1] but it more likely provides the support of the neutral meson. Now, the chamber photograph of the mixed shower shows frequently two or more cores of the soft shower starting from the shower origin. If this phenomena are thought to be due to the neutral meson, the mass of the neutral meson is determined by the simple analysis of the photograph. Those data obtained from the photograph are the number of the soft shower cores, its each energies and the angles between the cores. These situations are served as follows: 1) If the neutral meson is vector type, it decays into three photons. Let k, $\boldsymbol{k}_1$ k_2 $\boldsymbol{k}$ and k_3 $\boldsymbol{k}_3$ be the energy and impulse of these three photons in the laboratory system respectively, and let μ be the mass of the neutral meson. Then μ is determined from the simple calculation using Lorentz transformation. It results

$$\mu=[2(k_1k_2-\boldsymbol{k}_1\boldsymbol{k}_2)+2(k_2k_3-\boldsymbol{k}_2\boldsymbol{k}_3)+2(k_3k_1-\boldsymbol{k}_3\,\boldsymbol{k}_1)]^{1/2} \quad (1)$$

Let the neutral meson be scalar type, then it results putting $k_3\equiv 0$

$$\mu=[2(k_1k_2-\boldsymbol{k}_1\,\boldsymbol{k}_2)]^{1/2} \qquad (2)$$

ii) Though many data have been reported on the mixed shower, the experiments which give the precise value of the energy, angle, etc. of the soft shower cores seem not so abundant. Chao[2] gave the fine photographs of the mixed shower showing that the two soft cores start from the origin of the mixed shower. The energies and the angles of the cores are determined quite precisely. His results are not so reasonaly explained by the charge acceleration.[3] If we take alternatively the Chao's results as due to the neutral meson, we can determine the mass of neutral meson from Eq. (2). This result is given in Table (1). The mass of the neutral meson obtained from his data is much heavier than that of the neutral meson expected from nuclear force, —— about 1000 electron mass ——, except two. We know the particle of 1000 electron mass named τ-meson.[4] But τ-meson has been found so rarely that it is difficult to explain Chao's data from the view point of the neutral meson decay. It seems likely that the mixed shower detected by Chao is due to the more complex charge acceleration like processes or others. Recently Fretter[5] gave another data on the mixed shower. The energies of the soft cores in his paper, however, are not so precise unfortunately. We can onlo say that his data seem not inconsistent with the assumption of the decay of the neutral meson having 300 electron mass as its own mass.

Table 1

Chao's notation	G 1 No. 6	G 1 No. 7	G 1 No. 9	G 2 No. 10	G 2 No. 11	G 2 No. 12
μ electron mass unit	1000	1040	770	340	1040	440

1) S. Hayakawa, Phys. Rev. **75** (1949), 1759.
2) C. Y. Chao, Phys. Rev. **75** (1949), 581.
3) S. Hayakawa, Phys. Rev. **75** (1949), 1958.
4) Rochester & Butler, Nature **160** (1947), 855.
5) W. Fretter, Phys. Rev. **76** (1949), 511.

On the Integrability Condition of Tomonaga-Schwinger Equation.

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November 2, 1949

In several examples[1] it has been verified that the effect of the surface dependent additional Hamilton density function $g^2w\ (x, \sigma)$ (g is the coupling constant), which is introduced[2] in order to make the Tomonaga-Schwinger equation integrable, is just eliminated by the second order effect of the main Hamilton density $gH(x)$. In this note we shall give an outline of the argument on the general validity of such a cancellation and show that for the purpose of practical calculation, at least for the collision problem, one can dispense with the term $g^2w\ (x, \sigma)$. We hope that this fact will establish the equivalence[3] of Feynman's formalism[4] to that of Tomonaga and Schwinger in the more general cases of meson theories.

The fundamental equation of Tomonaga-Schwinger theory

$$i\hbar c\ \delta\Psi[\sigma]/\delta\sigma(x)=gH(x)\Psi[\sigma], \qquad (1)$$

may be solved by putting

$$\Psi[\sigma_F]=U[\sigma_F;\ \sigma_A]\ \Psi\ [\sigma_A], \qquad (2)$$

and by expanding U in powers of g:

$$U[\sigma_F;\ \sigma_A]=U^{(0)}+gU^{(1)}+g^2U^{(2)}+\cdots. \qquad (3)$$

These terms can be obtained successively, if we introduce a one-parametric family of space-like surfaces $\{\sigma_t\}$ between σ_A and σ_F (as Dyson[3] has done) so that one point x is passed by one and only one surface. For simplicty we shall hereafter confine ourselves to order g^2. Then we have

$$U=1+\left(\frac{-ig}{\hbar c}\right)\int_{\sigma_A}^{\sigma_F} H(x)d^4x+\frac{1}{2}\left(\frac{-ig}{\hbar c}\right)^2 \iint_{\sigma_A}^{\sigma F} P(H(x_1),\ H(x_2))d^4x_1d^4x_2. \qquad (4)$$

Now we must require that U should be independent of the special choice of the family of intermediate surfaces $\{\sigma_i\}$. The first two terms on the right hand side of (4) are obviously independent of $\{\sigma_i\}$. The third term can be expressed according to the definition of the P-symbol as

$$U^{(2)}=\frac{1}{2}\left(\frac{-i}{\hbar c}\right)\iint\Big[\frac{1+\varepsilon(x_1, x_2)}{2}\ H(x_1)$$

$$H(x_2)+\frac{1-\varepsilon(x_1, x_2)}{2}\ H(x_2)H(x_1)\Big]d^4x_1d^4x_2$$

$$=\frac{1}{4}\left(\frac{-i}{\hbar c}\right)^2\Big[\iint\{H(x_1), H(x_2)\}d^4x_1d^4x_2+$$

$$\iint \varepsilon(x_1, x_2)\ [H(x_1), H(x_2)]\ d^4x_1d^4x_2\Big] \quad (5)$$

where $\varepsilon\ (x_1, x_2)$ is equal to $+1$ or -1 according as the surface σ_1 (through x_1) lies in the future or the past side with respect to σ_2 (through x_2). The first integral again is not influenced by the positions of intermediate surfaces. In the second integral the factor $\varepsilon(x_1, x_2)$ is not altered by the choice of $\{\sigma_i\}$ as long as x_1 and x_2 lie time-like to each other, but if they are situated at space-like positions this factor becomes essentially dependent on the details of $\{\sigma_i\}$. Therefore, when $[H(x), H(x^1)]$ does not vanish for a pair of space-like points x and x', U is not independent of intermediate surfaces, and thus one is forced to supplement the Hamilton density $gH(x)$ with a term $g^2w(x, \sigma)$, which can be determined in the following manner, so as to compensate this unphysical surface-dependence of U.

Suppose we have added such a term $g^2w(x, \sigma)$ to $gH(x)$, then we have

$$U^{(2)}=\frac{1}{4}\left(\frac{-i}{\hbar c}\right)\Big[\iint\Big[\ \{H(x_1),\ \ H(x_2)\}$$

$$+\varepsilon(x_1, x_2)[H(x_1), H(x_2)]\Big]d^4x_1d^4x_2$$

$$+\left(\frac{-i}{\hbar c}\right)\int w(x_1, \sigma_1)d^4x_1, \quad (6)$$

and so

$$-\frac{i}{4\hbar c}\int\{\varepsilon(x_1, x_2)[H(x_1), H(x_2)]\}d^4x_2$$

$$+w(x_1, \sigma_1) \quad (7)$$

must be independent of $\{\sigma_i\}$. If we deform in (7) the surface σ_1 slightly at a point x_1' and take account of the realation

$$\frac{\delta\varepsilon(x_1, x_2)}{\delta\sigma(x_1')}=2\delta^4(x_2-x_1'), \quad (8)$$

we obtain the functional differential equation

$$\frac{\delta w(x_1, \sigma_1)}{\delta\sigma_1(x_1')}=\frac{i}{2\hbar c}\ [H(x_1), H(x_1')] \quad (9)$$

This is just the condition (to order g^2) derived by Kanesawa and Tomonaga[2] to determine the additional Hamiltonian.

From the above derivation it is quite clear that the addition and reelimination of $g^2w(x_1\sigma)$ is indeed an unnecessary detour in practice: one can simply use $gH(x)$ alone and strike out the surface-dependent terms from the result, since they are sure to be cancelled in the "rigorous" treatment.

To carry out this procedure, it will be convenient to introduce a modified P-symbol, P^* defined as follows:

$$P^*(A(x_1), B(x_2))=\frac{1+\varepsilon^*(x_1, x_2)}{2}\ A(x_1)$$

$$B(x_2)+\frac{1-\varepsilon^*(x_1, x_2)}{2}\ B(x_2)A(x_1). \quad (10)$$

where $\varepsilon^*(x_1, x_2)=1, 0, -1$, according as x_1 lies in the future time-like region, space like region, or past time-like region with regard to x_2. This will change Feynman's $\Delta_F=\Delta^{(1)}+i\varepsilon\Delta$ into $\Delta_F^*=\Delta^{(1)}+i\varepsilon^*\Delta$. ε^* is, contrary to ε, a Lorentz-invariant function and $\varepsilon^*\Delta$ differs from $\varepsilon\Delta=-2\Delta$ only when differentiated twice, because $\frac{\partial\varepsilon}{\partial x_4}\ \frac{\partial\Delta}{\partial x_4}=i\delta^4(x)$, while $\frac{\partial\varepsilon^*}{\partial x_4}\ \frac{\partial\Delta}{\partial x_4}=0$, and we need not take care of δ^4-function pointed out by Matthews[5] recently.

The analysis of higher order terms as well as of the effective value of an operator (radiative correction) will be treated in a fuller account that will appear later. In conclusion, we express our cordial thanks to Prof. Husimi and Dr. Utiyama for their interest taken in this problem and also to Mr. Mugibayashi, whose discussion has given impetus to these considerations.

1) *E.g.* F. J. Belinfante: Phys. Rev. **76** (1949), 66; K. M. Case: ibid. 14; Messieurs Fukuda and Miyamoto have pointed out this fact as early as in 1948 in their private conversation with us.
2) S. Kanesawa and S. Tomonaga: Prog. Theor. Phys. **3** (1948) 1, 101.
3) F. J. Dyson: Phys. Rev. **75** (1949) 486.
4) R. P. Feynman: Phys. Rev. **76** (1949) 769.
5) P. T. Matthews: Phys. Rev. **76** (1949) 684.

The Range and the Mean Free Path of High Energy Nucleons in Nuclear Matter.

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November 3, 1949.

Using the method due to Heisenberg and Bagge,[1] we have calculated the following quantities;

1) the range L,
2) the collision mean free path Λ, and
3) the cross section for energy loss

$$\sigma(E,\varepsilon)\approx\frac{1}{E}\ \varphi(\varepsilon)\times10^{-26}\mathrm{cm}^2$$

of a high energy nucleon in nuclear matter. E means the kinetic energy of the incident nucleon inside the nucleus and ε is its energy loss. Here we adopt the Fermi gas model for the nucleus and the following nuclear constants; the Fermi energy $E_F=22$ Mev. the depth of the nuclear potential hole $=30$ MeV and the nuclear radius $=R=1.37\ A^{1/3}\cdot10^{-13}$ cm.[2] Also, we used the central Yukawa potential determined by Serber et al, which can explain the 90 MeV neutron-proton scat-

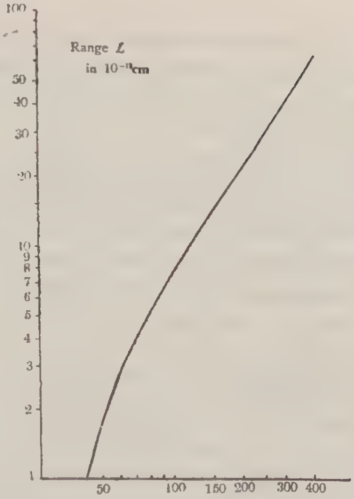


Fig. 1.

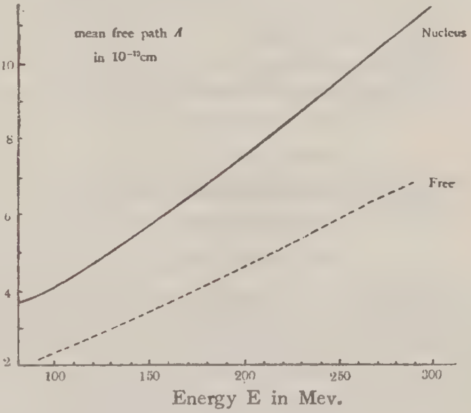
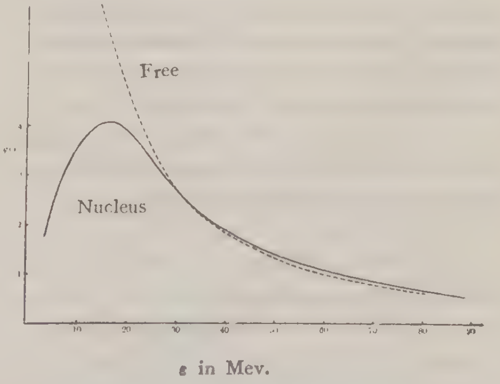


Fig. 2.



In Fig 2 and 3, "free" means the case in which one does not take account of Pauli's exclusion principle.

tering.³⁾ We have omitted a smaller part of the matrix elements for the scattering process, only in the case 3). The results are shown in Fig 1, 2, and 3.

On the basis of these results we can interpret the high energy nuclear experiments of Berkeley and the stars caused by Negative π-meson capture. The latter results have already been published in this Journal.⁴⁾

1) W. Heisenberg; Leipziger Beriger Berichte, **89** (1937), 369.
E. Bagge; Ann. d. Phys. **35** (1939), 431.
E. Williams; Nature, **142** (1938), 431.
Summary of these works is found in L. Rosenfeld; *Nuclear Forces*, vol. II (1949).
2) R. Serber et al; Phys. Rev. **75** (1949), 1352.
3) Cited in L. L. Foldy and E. Marshak; Phys. Rev. **75** (1947), 1498.
4) Y. Fujimoto, S. Hayakawa and Y. Yamaguchi; Prog. Theor. Phys. in press.

Notes on High Energy Nuclear Evaporation

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November 3, 1949.

Recently we have treated the problem of nuclear evaporation by generalizing the Bagge's method.¹⁾²⁾ In I, however, we have neglected the emission of nuclear fragments other than nucleons (neutrons and protons).

If we want to include the a-particle emission, the equations (5) and (6) in I must be replaced by the following formula:

$$-\frac{\partial x}{\partial a}=\frac{1}{\Gamma_n+\Gamma_p+4\Gamma_\alpha}\{(E_n+2T)\Gamma_n+(E_p+V+2T)\Gamma_p+(E_\alpha+2V+2T)\Gamma_\alpha\},\qquad(5')$$

$$a=n+p;$$

$$\frac{dn}{dp}=\frac{\Gamma_n+2\Gamma_\alpha}{\Gamma_p+2\Gamma_\alpha}\approx\frac{\Gamma_n}{\Gamma_P}\left(1-2\frac{\Gamma_\alpha}{\Gamma_P}\right);(6')$$

where n and p denote the total number of neutrons and protons lost from the initial nucleus (including the neutrons and protons evaporated in the form of a-particles) respectively. The a-particle width Γ_α is obtained from Γ_P by replacing $E_\cdot$ by

$$E_\alpha=2E_n+2E_P-(\textit{binding energy of } a\textit{-particle})\approx 4\text{Mev}\qquad(7')$$

and the Coulomb barrier height V by $2V$.*⁾

$$\Gamma_\alpha(x)\approx\frac{1}{2}\int_{2V+E_\alpha}^{x}(\xi-2V-E_\alpha)\times exp\,[S_{B'}(x-\xi)-S_A(x)]d\xi.\qquad(2')$$

$$S_{B'}(x)=0.362[(A-4)x]^{\frac{1}{2}}.\qquad(3')$$

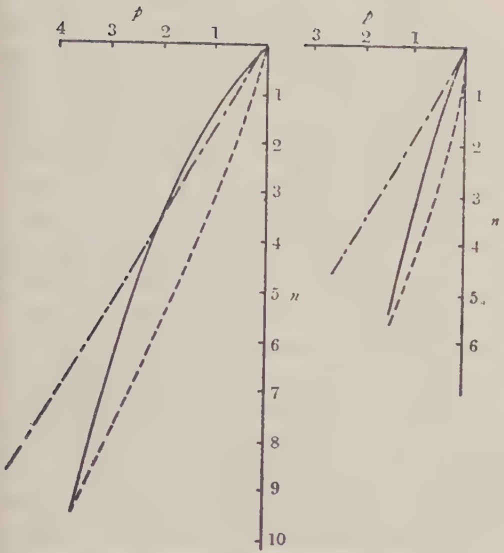
The factor 1/2 is due to the difference in statistical weights of nucleons (2) and a-particles (1). Solving the equations (5′) and (6′) with (7), (7′), we obtain the curves shown in Figs. 1 and 2 (case ($np\ a$)). For comparison, the solutions of (5), (6) with (7) are also shown in the same same figures by dotted curves (case (np)).

As is seen from these curves, the end points of the evaporation processes almost coincide for both the cases ($np\ a$) and (np). In general, we may roughly say for $A\lesssim100$, that the evaporation processes are taken place along the "Heisenberg valley" (i.e. the curve $n=\frac{1+\mu}{1-\mu}p$). Furthermore, the number of emitted nucleons $a=n+p$ is determined in the first approximation only by X and independent of the mass number A of the nucleus.

Table I shows the average numbers N_n, N_P and N_α of evaporated nentrons protons and a-particles for specified values of A and X. The ratio N_α/N_P is agreement with the experimental values of Perkins et al..³⁾

Table 1.
A=100

X in Mev	N_n	N_p	N_α
100	5.7	1.1	0.2
200	10.7_8	1.7_6	0.4_6

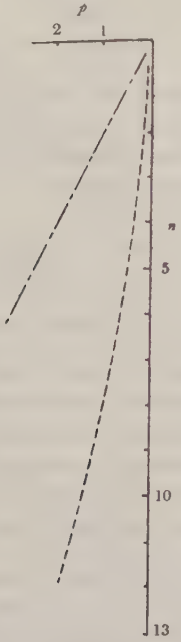


A=100
X=200 Mev

A=100
X=100Mev.

—·—·— Heisenberg valley
———— (*np a*)
· · · · (*np*)

Fig. 1.



A=200
X=200 Mev.
Fig. 2.

1) Y. Fujimoto and Y. Yamaguchi, Prog. Theor. Phys. in press, quoted as I. The notations here used are the same as in I.

2) E. Bagge, Ann. d. Phys., **39** (1941), 512.

*) α-particle width Γa is not argued in the present stage of nuclear physics. Here, however, we consider each α-particle as a unit of a nucleus. Let, for example, be $N=Z=$even number, in which case we may consider α-particles as a degenerate Bose gas in such a suitable potential hole that the total binding energy of a nucleus composed by α-particles can be explained. Then Weisskopf's evaporation theory can be applied, and we get approximately the same values for Γa as in the conventional formula (2).

3) D. H. Perkins et al., Proc. Roy. Soc. **196** (1949) 325; and private communication from Dr. Perkins.

Note on High Energy Induced Fission of Bi.

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November 3, 1949.

The high energy fission experiments : 190 MeV $d+\text{Bi}\rightarrow$fission have been performed by Goeckermann and Perlman.[1] In order to explain the mass number distribution of this fission products, these authors assumed that the evaporation process : $\text{Bi}^{209}(d, 12n)\,\text{Po}^{199*}$ occured at first, and then the residual nucleus Po^{199*} would undergo fission. Here we shall justify this explanation using the evaporation theory and Bohr-Wheeler's theory.

If we compare the emission width Γ_α of nuclear particle[2] (practically equal to the neutron width Γ_n) with the fission width Γ_f [3] in each step of evaporation process of the Bi nucleus with init excitation energy 190 MeV, we find Γ_f becomes larger than Γ_a after the nucleus evaporating about ten neutrons or more (and no or one proton).[4] Therefore, the residual nuclei Po* ($A\sim200$) or Bi* ($A\sim200$) undergo fission.

Of course, there exists in this bombardment initial excited nuclei with the excited energy lower than 190 MeV. But, since at such cases Γ_f can not become larger than Γ_α, the contributions from these lower initial excited nuclei should be considerably small, and may safely be but out of our consideration. Furthermore, the experimental total cross section for fission is approximately equal to the cross section for giving the whole deuteron energy Bi nucleus evaluated from the semi-transparent nucleus model.[5] This points, therefore, supports the fact that the main parts of fission are due to the 190 MeV Po^{211*}

Here Po* or Bi* ($A\sim 200$) have not so high excitation energies that a modification of Bohr-Wheeler's theory is of little use, as as mentioned in our separated paper.[6]

Thus we can justify the Goeckermann and Perlman's mechanism under the light of usual nuclear physics.

1) R. H. Goeckermann and I. Perlman ; Phys. Rev. **76** (1949), 628.
2) V. Weisskopf ; Phys. Rev. **52** (1937), 295.
Y. Fujimoto and Y. Yamaguchi ; Prog. Theor. Phys. in press.
3) N. Bohr and J. A. Wheeler ; Phys. Rev. **56** (1939), 426.
4) See the above letter.
5) R. Serber ; Phys. Rev. **72** (1947), 1114.
Also see Y. Fujimoto and Y. Yamaguchi ref. 2).
6) Y. Fujimoto and Y. Yamaguchi; Prog. Theor. Phys. in press.

On the Origin of Electronic Rays with Moderate High Energy.

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November 3, 1949.

The energetic electronic rays in the upper atmosphere, which are measured as bursts caused by the presence of a thin lead plate, are supposed to be originated mainly from high energy nuclear collisions. The various possible mechanisms to produce such radiation in a nuclear collision are proposed by several authors, among which most plausible ones are the neutral meson hypothesis and charge acceleration hypothesis. In order to test these hypothesis, we calculate the intensity of electronic rays under each hypothesis and compare it with the result of burst experiments.

(1) Neutral meson hypothesis.[1] We can unambiguously calculate the production spectrum of neutral mesons from the observed μ-meson spectrum,[3] under the usually adopted assumptions, (i) the altitude variation of the intensity of agent nucleons is $exp(-x/125)$, where x represents the atmospheric depth measured in g/cm², and (ii) the production rate of neutral mesons is a half of that of charged π-mesons. If a neutral meson immediately disintegrates into two photons, which is acceptable for its life shorter than 10^{-9} sec., the production spectrum per g. sec. of photons at the depth x' are given by

$$\gamma(x', \varepsilon)=4.6\times 10^{-6}(10BeV/\varepsilon)^2 \times exp\,(-x'/125), \quad (1)$$

where ε represents the energy of the photon. Multiplying by the cascade function, we get the vertical intensity per cm² sec of electronic component at the depth x ;

$$f(x, \varepsilon)=\int_0^x \gamma(x', \varepsilon)\pi(x-x')dx'=8.4\times 10^{-4} (10BeV/\varepsilon)^2\,\{exp\,(-x/125) -exp(-x/68)\}. \quad (2)$$

(2) Charge exchange hypothesis[2]. Hayakawa and Tomonaga calculated the cross section for the photon production due to the charge exchange of a nuclen in a nuclear collision.[4] Using this result, we obtain the production spectrum per g. sec. of photons

$$\gamma'(x', \varepsilon)=1.6\times 10k(10BeV/\varepsilon)^2 exp\,(-x'/125). \quad (3)$$

We see this is larger than (1) by the factor about $3k$.

To compare with experiments, we add

the contribution from the shielding matter, though a small fraction, and integrate over all directions. The intensity of electronic rays with the energy greater than 10 BeV is shown in Fig. 1. In comparison, we plot the experimental value of bursts measured by Bridge and Rossi,[5] and Hulsizer and Rossi.[6] Some inconsistency seems to be between both experiments, although we adopt the very values taken by the authors. If we refer to Bridge and Rossi, the charge exchange process has so little effect that k should be as small as a fraction of unity, while, referring to Hulsizer and Rossi, we must take into account the charge exchange process over γ-decay, so that k should be nearly equal to unity. The precise determination of the absolute intensity of energetic electronic rays will be hoped for answer our question.

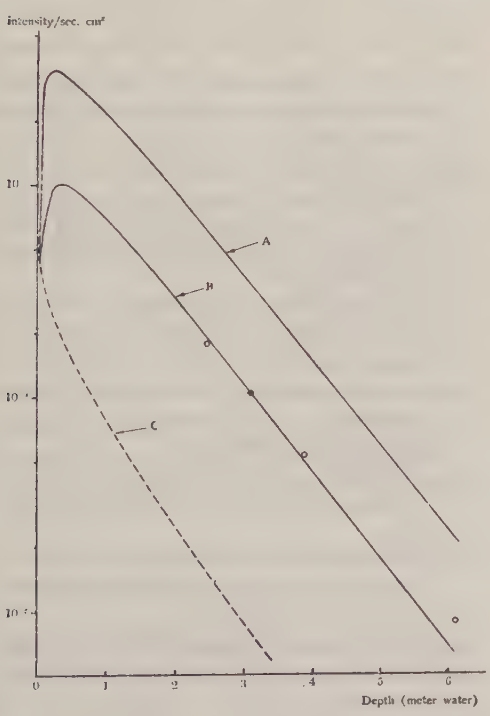

Fig. 1. Total intensity of electronic component with energy greater than 10 BeV. Calculated curves; A... charge exchange hypothesis, B... neutral meson hypothesis, C... contribution from shielding matter (A and B include contribution C). Experimental points; X...Hulsizer and Rossi, o...Bridge and Rossi.

1) H. W. Lewis, J. R. Oppenheimer and S. A. Wouthuysen; Phys. Rev. **73** (1948), 127.
K. Greisen; Phys. Rev. **73** (1948), 521.
2) S. Hayakawa; Phys. Rev. **75** (1949), 1759
3) S. Hayakawa and J. Nishimura; Prog. Theor. Phys. **4** (1949), 232. and ibid in press.
4) S. Hayakawa and S. Tomonaga; Prog. Theor. Phys. **2** (1947), 162.
5) H. Bridge and B. Rossi; Phys. Rev. **71** (1947), 378.
B. Rossi; Rev. Mod. Phys. **21** (1949), 104.
6) R. I. Hulsizer and B. Rossi; Phys. Rev. **73** (1948), 1402.

Note on Varitrons

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November 3, 1949

By a series of elaborate experiments, Alikhanov and Alikhanian et al[1] have discovered many variety of cosmic ray particles called varitrons. Their results seem us to be certain, although the scattering effect in the lead plate placed above the third tray in their magnetic analizer is not taken into account, because of the cosistency among various results obtained by different experimental methods.[2,3]

In the magnetic analizer and the cloud chamber system of Brode,[3] varitrons fly through long paths ($>1m$) without disintegrations. This fact means most of varitrons have long lives, which is consistent with the delayed coincidence experiment[4] and the existence of maximum momenta in momentum spectrum[5]. Furthermore, the tracks of varitrons in photographic plates are not associated by stars in their track ends. These facts seem to argue that most of varitrons may not directly interact with nucleous and be cousins of mu-mesons. On the contrary, there is an evidence that varitrons are likely produced by nuclear interactions.[6] The directly produced varitrons, the brothers of pi-mesons, are supposed to immediately disintegrate into descendant ones, the latter being usually observed. This view will be supported by the existence of varitrons lighter than pi-mesons, as suggested by Taketani.[7] He pointed out that such lighter varitrons should be observed in Berkeley experiments if they would couple strongly enough with nucleous. Then the coupling of varitrons with their ancesters may be as large as to correspand to the rather short lives of the ancesters and as small as to be forbidden the transition to a lighter nuclear meson, presumably pi-meson, The masses of partners coupling with a common ancester should not be so great as to give rise to appreciable events of nuclear disintegrations.

The existence of varitrons with rather long lives as mu-mesons would affect the meson life measured by delayed coincidence method Ticho[8] obtained at mountain elevation too short life of mu-mesons, which can not be explained even by accounting for the capture of negative mu-mesons. It is likely that he measured the lives of varitrons in addition to mu-mesons.

In the photographic plates there are some long tracks with the mass about 700 times electron mass starting from stars. This sort of varitron should directly couple with nucleons and is supposed to be the same kind as 'the another tau-meson'[9]. This is a more argument for this kind of meson to be distinguished from tau-meson because of the absence of stars in their track ends.

A disintegration of 800 m varitron into 200 m one[10] is similar to that of $\tau\to 3\pi$ decay discovered by Bristol group,[11] considering the values of masses may have some tolerance due to the unsatisfactory accuracy of this experiment. If it is the case, the disintegrated pi-meson should be positive because it does not produce a star, but there is not seen a secondary mu-meson, probably because of the low sensitivity of the plate. In the presence of many sorts of varitrons the disintegrations of Rochester and Butler[12] are not necessary to ascribe to $\tau\to\pi+\pi^0$ or γ decay. Thus the difficulty in the competition between two and three particle decays can be avoided. Such direct coupling varitrons may rarely occur in experiments but slightly affect the nuclear foeces.

It is not ruled out that some sorts of varitrons disintegrate into electrons. Then some of Bruxelles decays[13] are not unlikely played by varitrons. If it were the case abnormal beta-spectra remarked by Nakamura and Ono[14] could be explained by the decay electrons cf varitrons, which would directly couple with the nucleous.

Detailed account will be appeared elsewhere.

1) c. f. A. I. Alikhanian, A. I. Alikhanov and A. O. Vaisenberg: J. Exp. Theor. Phys. **18** (1948), 301.

2) A. I. Alikhanian, D. M. Samoilovich, I. I. Gurevich, Kh. P. Babayan and R. I. Gerasimova: J. Exp. Theor. Phys. **19** (1949), 664.

3) J. G. Retalack and R. B. Brode: Phys. Rev. **75** (1949), 1716.
R. B. Brode: Rev. Mod. Phys. **21** (1949), 37.

4) A. O. Vaisedberg: J. Exp. Theor. Phys. **19** (1949), 726.

5) A. I. Alikhanian, A. I. Aleikhanov, V. M. Mornzov, and A. V. Khrimiad: C. R. Acad. Sci. USSR, **51** (1948), 39.

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The Decay of a $\tau^{\pm}$ Meson into $\pi^{\pm}$ and π_0 Meson.

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November 6, 1949.

Calculation of the decay life-time of a $\tau^{\pm}$ meson into $\pi^{\pm}$ and π_0 mesons has been performed in order to compare it with that of a τ meson into 3π mesons, which has been found in the photographic emulsion in America.[1] Assuming that $\tau^{\pm}$, $\pi^{\pm}$ and π_0 mesons are all Bose particles and decay through virtual creation of nucleon parir, we examined all possible cases of meson types and couplings in both symmetrical and neutral theories. Although the results are ambiguous in consequence of the pathological charactor of delta function of Jordan and Pauli, we exclude this ambiguity by Pauli's "regulator",[2] but sacrifying some properties of matrix-element such as Dyson's equivalence theorem of the pseudoscalar meson.[3] The results will give us the upper estimate of the lifetime in the correct future theory, so we may deduce some conclusion from the ambiguous results of the present field theory.

The matrixelement of this process in Tomonaga-Schwinger theory is

$$M=\frac{(1+(-1)^{N})}{2}\int dx'dx''\mathrm{Spur}\{\bar{S}(x-x')\gamma_{\pi_{\pm}}$$
$$\bar{S}(x'-x'')\gamma_{\pi 0}\ S_1(x''-x)\gamma_{\tau}+etc\}U'_{\pi_{\pm}}U_{\pi 0}''U_{\tau} \qquad \text{(I)}$$

where $U_{\pi\pm}$($U_{\pi 0}$ or U_{τ}) is the potential or the some derivatives of potential describing $\pi^{\pm}$ (π_0 or τ) meson field. The spin matrix γ_{π}($\gamma_{\pi 0}$ or γ_{τ}) is respectively 1, γ_i, γ_{ij}, γ_{ijk} γ_5 (γ being Dirac matrices) according to $U_{\pi\pm}$ ($U_{\pi 0}$ or U_{τ}) is a scalar, vector, tensor, pseudovector or pseudoscalar. N is the sum of the number of vector couplings and tensor couplings and τ_3 isotopic spin occuring in (I).

Before carrying through the evaluation of this matrixelement, we derive some properties of it, because the singular delta function could make the result ambiguous. (1) The generalized Furry's theorem,[5] i. e. if N is an odd number, the process is forbidden. From this theorem, the process is forbidden either in symmetrical theory or in neutral theory. (2) Lorentz invariance of the matrixelement. For example, if the τ meson and π mesons are all pseudoscalar mesons, a scalar quantity cannot be made from these three pseudoscalar potentials and two independent propagation vectors of these mesons, therefore the process is forbidden in this case. (3) The vector coupling of a scalar meson in neutral theory is always forbidden. This can be easily shown by using the following nature of $\bar{S}$, S_1 functions and Furry's theorem.

$$\left(\gamma\ \frac{\partial}{\partial x}+\varkappa\right)\ \bar{S}(x)=-\delta(x)$$
$$\left(\gamma\ \frac{\partial}{\partial x}+\varkappa\ \right)S_1(x)=0 \qquad \text{(II)}$$

(4) Equivalence between pseudovector coupling and pseudoscalar coupling of the pseudoscalar meson field is true in some cases, which can be shown also by using the properties (II) and others. (5) The symmetry of the matrixelement (I) with respect to τ, $\pi^{\pm}$ and π_0 mesons. Using this property, the life of those decay process, in which we interchange the types and couplings of τ meson with those of $\pi_{\pm}(\pi_0)$ meson, is only derived from the former life by interchanging the mass of τ meson with the mass of $\pi_{\pm}(\pi_0)$ meson.

The matrix element is evaluated as follows. Inserting in (I)

$$\bar{S}(x)=\frac{1}{(2\pi)^4}\int dk\ \frac{(\gamma_i\ k-\varkappa)}{k^2+\varkappa^2}\quad e^{iKX}$$
$$S_1(x)=\frac{1}{(2\pi)^3}\int dk(\gamma ik-\varkappa)\delta(k^2+\varkappa^2)e^{iKX}$$

and integrate with respect to $dx'\ dx''$, we obtain

$$M=\frac{(1+(-1)^N)}{2(4\pi)^{11}}\int dkdk'dk''\delta(k-k'-P_\pi)$$

$$\delta\ (k'-k''-P_{\pi0})\int_0^1\frac{udu}{2}\int_{-1}^1 dr\delta''(\varkappa^2+$$

$$\xi k^2+\xi_1k'^2+\xi_2k''^2)\times\mathrm{Spur}\{\gamma ik-\varkappa)\gamma_\pi\,(\gamma ik'-\varkappa)\gamma_{\pi0}(\gamma ik''-\varkappa)\gamma_\tau\}U_\pi U_{\pi0}U_\tau$$

where

$$(\xi.\ \xi_1,\ \xi_2)=\left(\frac{u(1+v)}{2},\ 1-u,\ \frac{u(1-v)}{2}\right)$$

$$\xi+\xi_1+\xi_2=1$$

and changing the variables

$$k=\bar{k}+\xi_1P_\pi+\xi_2P_\tau,\quad k'=\bar{k}+\xi_2P_{\pi0}-\xi P_\pi,$$

$$k''=k-\xi P_\tau-\xi_1P_{\pi0}$$

and integrate with respect to $d\tilde{k}$ accordig to

$$\int d\tilde{k}\,\delta''(k^2+A)=\frac{\pi}{A},\quad\int d\tilde{k}(\tilde{k}^2+A^2)\delta''(\tilde{k}^2+A)$$

$$=2\pi\lim_{k\to\infty}\left\{\log\frac{(k^2+\sqrt{k^2+A}\ )^{\mathrm{I}}}{A}-2\right\}$$

Finally, we expand the result in power series of the ratio of the meson mass to the nucleon mass, and integrate over du, dv according to

$$\int_0^1\frac{udu}{2}\int_{-1}^1 dv\ \xi^\alpha\ \xi_1{}^\beta\ \xi^\gamma{}_2$$

$$=\frac{\alpha!\beta!\gamma!}{(\alpha+\beta+\gamma+2)!}$$

The evaluated matrix element has sometimes logarithmic divergence, and sometimes has the ambiguity, which contradicts to the properties (3) (4) of the mâtrixelement. We exclude these divergency and ambiguity by Pauli's "regulator"

$$\int\rho(m)dm=0\qquad\int\rho(m)\log|m|dm=0$$

$$\int m\rho(m)dm=0\qquad\int\rho(m)\log|m|mdm=0$$

but sacrifying some properties such as Dyson's equivalence theorem of pseudoscalar meson field, which will give us the upper estimate of lifetime in correct future theory.

The decay lifetime of a $\tau^\pm$ meson into a $\pi^\pm$ and a π_0 meson, if it is not forbidden, lies in general $10^{-14}\sim10^{-16}$ sec., by using the next value of meson masses and coupling constants

$$\mu_\tau=900m_e\qquad\mu_\pi=\mu_{\pi0}=300m_e$$

$$\frac{g_\pi{}^2}{\hbar c}=\frac{g_{\pi0}{}^2}{\hbar c}=\frac{1}{10}\qquad\frac{g_\tau{}^2}{\hbar c}=\frac{1}{10^2}$$

The life of this process is too shorter than that of 3π decay. This process must be forbidden, so that the τ meson does not decay to π and π_0 mesons before 3π decay. Detailed account will be published elsewhere.

In conclusion, the authors wish to express their coordial thanks to Prof. Tomonaga for his kind encouragement in this work, and also to Dr. Ozaki and Mr. Hayakawa for their helpfull discussions.

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The Decay of a $\tau^\pm$ Meson into a $\pi^\pm$ Meson and a Photon.

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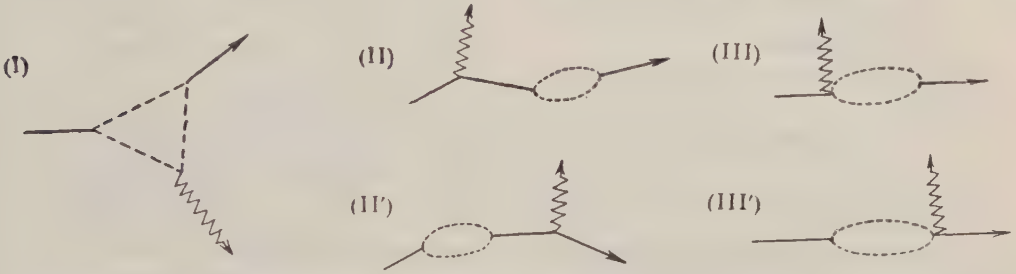
Calculation of the decay lifetime of a $\tau^\pm$ meson into a $\pi^\pm$ meson and a photon must be examined before evaluating the life of the 3π decay. Since this decay process is lower order one than the 3π decay, and, moreover, there is no such a selection rule due to Furry's theorem[1] as in 2π decay above-mentioned. The matrixelelement of this process consists of three types which can described as follows in Feynman's diagram.[2]



The type III (III′) is not necessary, when τ (π) meson is a scalar or pseudoscalar meson and its coupling is a scalar or pseudoscalar meson and its coupling is a scalar or pseudoscalar type. The gauge invariance of this matrix element is established only considering these three types altogether. But, when the coupling of the $\tau(\pi)$ meson is a vector or tensor type, and the coupling of the π (τ) meson is a scalar, pseudoscalar or pseudovector sype, the processes II, II′, III and III′ have no contribution to the matrixelement according to Furry's theorem, so the gauge invariance is established only by (I) in this case.

The matrix element has several important properties. (1) There is no Furry's theorem, because the isotopic spin τ_P does not commute or anticomute with the isotopic spin τ_{PN} and τ_{NP}. (2) The Lorentz and gauge invariance of the matrix element. For example, if τ and π mesons are both scalar (pseudoscalar) mesons, a gauge invariant scalar quantity cannot be made from these two scalar (pseudoscalar) potentials, the vector potential describing the electromagnetic field, and two independent propagation vectors of these mesons and a photon, therefore this process is forbidden. (3) When there is the vector coupling of a scalar meson the process is forbidden except that the coupling of another meson is a vector or tensor type. But these exceptional cases are also forbidden by the condition of zero photon mass, which has been formaly proved by Schwinger,[3] for the former case, and by using Pauli's "regulator" for the latter respectively. (4) The equivalence theorem of pseudoscalar meson[5] is maintained except that the coupling of another meson is a vector or tensor type, but in these cases it is also established by using "regulator". (5) The symmetry of the matrix element between τ and π meson.

We evaluated this matrix element by using Tomonaga-Schwinger's theory,[6] which is the same procedure as the above-mentioned letter. The evaluated matrixelement has sometimes the logarithmic divergency and sometimes the ambiguity, which are contradicted to the properties (3) and (4) of the matrix element. We removed these difficulties using Pauli's regulator but sacrifying the equivalence theorem of a pseudoscalar meson field. This procedure will give us the upper estimate of the lifetime of this decay, because the divergence and ambiguity in this calculation can make the lifetime shorter.

The decay lifetime of a τ meson into a π meson and a photon lies between $10^{-14} \sim 10^{-13}$ sec. in general, if this process does not forbidden, which is longer than that of above-mentioned 2π decay due to the smallness of coupling constant of the electromagnetic field with nucleon as comparison with that of meson with nucleon, but it is shorter than the lifetime of 3π decay.

We select the types and couplings of τ and π mesons, so that the τ meson does not decay into a π meson and a photon before into 3π decay, and that the 3π decay does not forbidden. The decay of a τ meson into π and π_0 mesons can be always forbidden by choosing the theory either symmetrical or neutral theory. The selected cases are listed in the below table assuming the types and couplings with nucleon of these three π mesons are all the same one.

τ	type	S	,,	,,	,,	,,	V	,,	Pv	,,	Ps	,,	,,	,,
	coupling	s	V	,,	,,	,,	V	t	t	,,	Ps	,,	Pv	,,
π	type	S	,,	V	,,	Pv	S	,,	,,	Pv	Ps	,,	,,	,,
	coupling	s	V	V	t	t	V	,,	,,	t	Ps	Pv	Ps	Pv
Theory		Sy	Sy N	Sy N	Sy N	Sy N	N	N	Sy N	Sy N	Sy N	Sy N	Sy N	Sy N
Convergency		∞	∞	∞			∞			∞	∞			∞

The suffix Sy and N in the table means that the theory is the symmetrical and neutral one respectively, and it is so choosen as to forbid the 2π decay. When the matrix element of the 3π decay has logarithmic divergence it is suffixed ∞. Detailed account will be published elsewhere.

The authors express their coordial thanks to Prof. Tomonaga for his kind encouragement in this work and also to Dr. Ozaki and Mr. Hayakawa for their helpful discussions.

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3) J. Schwinger Phys. Rev. **74** (1948), 1438.
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7) J. Schwinger; Phys. Rev. **74** (1948), 1438 **75** (1949), 347.

On the Electromagnetic Properties of Nucleons*

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November 14, 1949.

We have calaulated, using Tomonaga-Schwinger's method of canonical transformation, how the electromagnetic properties of a nucleon is altered by the scalar or pseudoscalar meson field surrounding the nucleon, up to the orders of $ef.^2$ We have omitted *DSS*-terms in the integrand, in which the arguments do not circulate (*D* and *S* are the 4-dimentional delta-functions of meson and nucleon respectively, and *DSS* means the product of one meson delta-function and two nucleon delta functions), interpreting them as mass renormalization terms according to Dyson's discussions.1) The terms which can be interpreted as vacuum polarization effects of nucleon field vanish exactly. Hence, there remain only Lamb shift type corrections. And the divergencies which arise from these terms can be subtracted as charge renormalization effects for nucleons. Then, the results become finite, and can be compared with experiments.

Considering various kinds of external electromagnetic fields, we have calculated the anomalous magnetic moment of a nucleon, interaction energy between a neutron and an electron, and the contribution to level shift of hydrogen atom.

Results obtained coincide almost with Case s results.(2) Therefore we give only results briefly:

A) Anomalous Magnetic Moments
I) Scalar Theory
a) neutral theory

$$\mu_p=\frac{f^2}{4\pi\hbar c}\cdot\frac{1}{2\pi\lambda^2}[E_3(\lambda^2)-2E_2(\lambda^2)]=$$
$$\frac{f^2}{4\pi\hbar c}\cdot\frac{1}{2\pi}\Big[\frac{3}{2}-\lambda^2+\lambda^2(3-\lambda^2)\ \log\ \frac{1}{\lambda}$$
$$-\frac{\lambda(4-5\lambda^2+\lambda^4)}{\sqrt{4-\lambda^2}}\ \cos^{-1}\frac{\lambda}{2}\Big],\quad \mu_N=0$$

b) charged theory

$$\mu_P=\frac{f^2}{4\pi\hbar c}\cdot\frac{1}{\pi}\Big[-F_1(\lambda^2)+F_3(\lambda^2)\Big]$$
$$=\frac{f^2}{4\pi\hbar c}\cdot\frac{1}{\pi}\Big[\frac{5}{2}-\lambda^2-(2-4\lambda^2+\lambda^4)\ \log\ \frac{1}{\lambda}$$
$$-\frac{\lambda(8-\lambda^2+\lambda^4)}{\sqrt{4-\lambda^2}}\ \cos^{-1}\frac{\lambda}{2}\Big],$$
$$\mu_N=-\mu_p{}^{(Charged)}$$
$$+2\mu_p{}^{(neutral)}=\frac{f^2}{4\pi\hbar c}\cdot\frac{1}{\pi}\Big[-1+(2-\lambda^2)$$
$$\log\ \frac{1}{\lambda}+\frac{\lambda(4-\lambda^2)}{\sqrt{4-\lambda^2}}\ \cos^{-1}\ \frac{\lambda}{2}\ .$$

II) Pseudoscalar theory

a) neutral theory

$$\mu_P=\frac{f^2}{4\pi\hbar c}\cdot\frac{1}{2\pi\lambda^2}[-E_3(\lambda^2)]=\frac{f^2}{4\pi\hbar c}\cdot\frac{1}{2\pi}$$
$$\Big[-\frac{1}{2}-\lambda^2+\lambda^2(1-\lambda^2)\ \log\ \frac{1}{\lambda}+\frac{\lambda^3(3-\lambda^2)}{\sqrt{4-\lambda^2}}$$
$$\cos^{-1}\frac{\lambda}{2}\Big],\qquad \mu_N=0.$$

b) charged theory

$$\mu_N=\frac{f^2}{4\pi\hbar c}\cdot\frac{1}{\pi}\Big[F_1(\lambda^2)-2F_2(\lambda^2)+F_3(\lambda^2)\Big]$$
$$=\frac{f^2}{4\pi\hbar c}\cdot\frac{1}{\pi}\Big[\frac{1}{2}-\lambda^2+\lambda^2(2-\lambda^2)\ \log\ \frac{1}{\lambda}-$$
$$\frac{\lambda(2-4\lambda^2+\lambda^4)}{\sqrt{4-\lambda^2}}\ \cos^{-1}\ \frac{\lambda}{2}\Big]$$
$$\mu_N=-\mu_P{}^{(charged)}\ +2\mu_P{}^{(neutral)}\ =\frac{f^2}{4\pi\hbar c}\cdot$$
$$\frac{1}{\pi}\Big[-1-\lambda^2\ \log\ \frac{1}{\lambda}+\frac{\lambda(2-\lambda^2)}{\sqrt{4-\lambda^2}}\cos^{-1}\frac{\lambda}{2}\Big],$$

where

$$E_n(\lambda^2)=\int_1^\infty\frac{du}{u^n(u^2-u+\frac{1}{\lambda^2})}\quad,$$
$$E_n^{(2)}(\lambda^2)=\int_1^\infty\frac{du}{u^n(u^2-u+\frac{1}{\lambda^2})}\ ,$$
$$F_n(\lambda^2)=\int_1^\infty\frac{du}{u^n(u^2+(\lambda^2-2)u+1)}\quad,$$
$$F_n(\lambda^2)=\int_1^\infty\frac{du}{u^n(u^2+(\lambda^2-2)u+1)^2}$$

and $\lambda=\frac{\mu}{k_0}$ (μ=meson mass,

k_0=nucleon mass).

B) Neutron-Electron Potential [divided by $\frac{4\pi r_0{}^3}{3}$: r_0=election radius]

I) Scalar Charged Theory $(\frac{f^2}{4\pi\hbar c}=3\lambda)$

$$V_0=\frac{f^2}{4\pi\hbar c}\cdot\frac{1}{\pi}\Big[-\frac{1}{6}F_3(\lambda^2)-\frac{1}{3}(F_1{}^{(2)}(\lambda^2)-$$
$$F_3{}^{(2)}(\lambda^2))-\frac{1}{48}E_3(\lambda^2)-\frac{1}{96}E_3{}^{(2)}(\lambda^2)+\frac{1}{6}$$
$$(E_1{}^{(2)}(\lambda^2)-E_3{}^{(2)}\ (\lambda^2))\Big]\int\frac{3e^2A_\mu}{4\pi r_0{}^3k_0{}^2}\Box^2 j_\mu d\tau$$
$$\doteqdot\frac{f^2}{4\pi\hbar c}\cdot\frac{1}{\pi}\Big[\frac{5}{2}\ \log\ \lambda-\frac{5}{12}\ -\frac{1}{3\lambda^2}$$
$$+\frac{5\pi}{12\lambda}\int\frac{3e^2A_\mu}{4\pi r_0{}^3k_0{}^2}\Box^2 j_\mu\ d\tau$$
$$\doteqdot\begin{cases}-10.0\ \ \text{KeV} & (\lambda=\frac{1}{10})\\ -\ 5.5\ \ \text{KeV} & (\lambda=\frac{1}{6})\\ -\ 1.3\ \ \text{KeV} & (\lambda=\frac{1}{2})\end{cases}$$

II) Pseudoscalar Charged Theory $(\frac{f^2}{4\pi\hbar c}=6)$

$$V_0=\frac{f^2}{4\pi\hbar c}\cdot\frac{1}{\pi}\Big[-\frac{1}{6}F_3(\lambda^2)+\frac{1}{3}\ (F_1{}^{(2)}(\lambda^2)$$
$$-2F_3(\lambda^2)+F_3{}^{(2)}(\lambda^2))-\frac{1}{48}E_3(\lambda^2)+\frac{1}{24}\ E_3{}^{(2)}$$
$$(\lambda^2)\Big)\Big]\int\frac{3e^2A_\mu}{4\pi r_0{}^3k_0{}^2}\Box^2 j_\mu d\tau\doteqdot\frac{f^2}{4\pi\hbar c}\cdot\frac{1}{\pi}$$
$$\Big[\frac{1}{2}\ \log\ \lambda+\frac{5}{6}-\frac{3\lambda\pi}{4}\Big]\int\frac{3e^2A_\mu}{4\pi r_0{}^3k_0{}^2}\Box^2 j_n d\tau$$
$$\doteqdot\begin{cases}-3.5\ \ \text{KeV} & (\lambda=\frac{1}{10})\\ -2.9\ \ \text{KeV} & (\lambda=\frac{1}{6})\end{cases}$$

C) Contribution to Hydrogen S-level Shift (δE)

This value is nearly proportional to the value of neutron-electron potential, for the contributions from nucleon charge are negligible compared to those from meson charge, and will show only the numerical results.

I) Scalar Charged Theory $(\frac{f^2}{4\pi\hbar c}=3\lambda)$

$$\delta E \doteq \begin{cases} 2.1 \quad Mc & (\lambda=\frac{1}{10}) \\ 1.2 \quad Mc & (\lambda=\frac{1}{6}) \\ 0.27 \quad Mc & (\lambda=\frac{1}{2}) \end{cases}$$

II) Pseudoscalar Charged Theory $(\frac{f^2}{4\pi\hbar c}=6)$

$$\delta E \doteq \begin{cases} 0.72 \quad Mc & (\lambda=\frac{1}{10}) \\ 0.50 \quad Mc & (\lambda=\frac{1}{6}) \end{cases}$$

*) These results were published in "Soryushiron Kenkyu" Vol. I (1949) 89, 307 (in Japanese). After one of these papers was completed, the authors read papers by Slotnick and Heitler(3) and by Case.(2)

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3) Slotnick & Heitler, Phys. Rev. **75** (1949), 1945.

The Covariant Formalism of the Theory of Damping

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In spite of its brilliant success in quantum electrodynamics, the theory developed by Tomonaga, Schwinger and Feynman (T. S. F. theory) cannot immediately be applicable to the processes involving meson fields, because of thier divergent nature in the high energy region. We should therefore resort to the damping theory proposed by Heitler and his collaborators,(1) in order to obtain the reasonable results. Thier prescription as it was is though insufficient in that the theory is given only in non-relativistic way and, as the consequence, under the assumption that in all the reaction terms are to be omitted correspondence argument, though the dropped terms give rise to ultra-violet difficulties if calculated non-relativistically.(2) moreover, the latter procedure has the serious defects, because, as was pointed out by Bethe and Oppenheimer,(3) it can do nothing about the well known infra-red catastrophe. Therefore, it is desirable to derive the relativistically covariant formalism for the damping theory. This paper deals with such a derivation.

It has been already shown by one of us(4) that the unitary operater in T. S. F. theory can also be expressed in the form of Cayley transformation

$$S[\sigma]=1-2\pi i R=\frac{1-2\pi i K}{1+2\pi i K}. \qquad (1)$$

Here, if we push the surface σ to infinite future, the expression for Heisenberg's S-matrix will result(5) and the corresponding K is calculated to be

$$K=\Sigma K^{(n)}, \quad K^{(n)}=\frac{1}{\pi 2^n}(-i)^n\int\cdots\int dx_1 \cdots dx H(1)\cdots H(n)\varepsilon(1,2)\cdots\varepsilon((n-1)n). \qquad (2)$$

or, as the explicit form of $S=\Sigma S^{(n)}$ have already been given elsewhere, it will also be convenient to have the relation between $S^{(n)}$ and $K^{(n)}$. From Eq. (1), we have

$$(1+\pi i K)S=(1-\pi i K),$$

and, by substituting $\Sigma K^{(n)}$ for K and $\Sigma S^{(n)}$ for S, and comparing both sides,

$$K^{(1)}=0,$$

$$K^{(2)}=-\frac{i}{4\pi}\iint_{-\infty}^{\infty} dx_1 dx_2 P(H(1)H(2))$$

$$=\frac{i}{2\pi}\cdot S^{(1)},$$

$$K^{(3)}=-\frac{1}{12\pi}\iiint_{-\infty}^{\infty}dx_1dx_2dx_3P(H(1)H(2)$$

$$H(3))=\frac{i}{2\pi}\ S^{(2)}, \qquad (3)$$

$$K^{(4)}=\frac{i}{2\pi}S^{(4)}-\frac{i}{4\pi}[S^{(2)}]^2,$$

Now, the operator R in Eq. (1) relates directly to the cross section σ_{AB} in the scattering processes through :

$$\sigma_{AB}=\frac{2\pi}{\hbar\sigma_A}\mid R_{AB}\mid{}^2\rho_B \qquad (4)$$

where A and B denote initial and final states, respectively, and V_B the velocity of the incident particle : ρ_B, the level density of the final states. The relation between K and R is obtained from Eq. (1) as

$$R=K-i\pi KR, \qquad (5)$$

which is nothing but the Heitler's equation generalised by Pauli and Wentzel. The only difference is that K has now the covariant form, including reactive effects in itself, and that the diverging effects due to the radiative mass and vacuum polarisation are previously amalgamated. If solved by successive substitution, Eq. (5) will give rise to the usual perturbational expantion, which coincides with that of T. S. F. theory. To have the damping effect, however, we have only to treat it as simultaneous equations. The applications to the individual scattering processes are now on calculation.

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A Remark on the Problem of Gauge-Invariancy

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December 6, 1949.

Previously, we pointed out that the self-energies W of a photon with momentum $l^{\sim}$ due to the various charged fields have the same relation of signs and proportionarities as those between zero-point energies E_{zero} of these charged fields ; that is, by the calculation in momentum space, we obtain;[1]

$$W=\frac{ne^2}{3\pi l}\left\{\frac{1}{2}w-\mu^2\right\}_{w\to\infty} \quad (l=|l|) \qquad (1)$$

$$n=\begin{cases}+\ 1 \text{ for scalar charged field}\\ -\ 2 \text{ for spinor charged field}\\ +\ 3 \text{ for vector charged field}\end{cases}$$

whereas

$$E_{zero}=n\int(p^2+\mu^2)^{\frac{1}{2}}\frac{d\boldsymbol{P}}{(2\pi)^3}\ . \qquad (1)'$$

It is our purpose to scrutinize the physical meanings of this esthetic relation from the correspondence theoretical view point. In the Dirac[2] Heisenberg's[3] theory of the positron using the density matrix, the following singular term is defined as the vacuum value of the density matrix which should be subtracted in the course of the calculation of vacuum polarization ;

$$(x'|S|\ x'')=u\frac{d^\rho\ x_\rho}{(x^\lambda x_\lambda)^2}-\frac{v}{(x^\lambda x_\lambda)}+w\ \log\frac{|x^\lambda x_\lambda|}{C}, \qquad (2)$$

$$u=-\frac{i}{2\pi^2}\ \exp\left(-ie\int_{x'}^{x''}A_\lambda dx_\lambda\right)\ . \qquad (2')$$

The factor u in (2) enables this subtraction theory to give the gauge invariant results.[4] According to Weisskopf,[5] this subtraction prescription is equivalent to define the self-energy of the vacuum as follows :

$$U^{(e)}=-\frac{2}{(2\pi)^3}\int\left\{\sqrt{(\boldsymbol{P}-e\boldsymbol{A})^2+\mu^2}-eA_0\right\}d\boldsymbol{P}. \qquad (3)$$

Therefore, the following postulate can be introduced ; " we should subtract as the self-energy of the vacuum the modified zero-point energy in which the momentum P_μ in the usual zero-point energy is replaced by $p_\mu - eA_\mu$". In the classical theory, as$\{\sqrt{(\boldsymbol{P}-e\boldsymbol{A})^2+\mu^2}-eA_0\}$ is $\mu/\sqrt{1-v^2}$, the above definition of the vacuum corresponds to specification of the state of electrons by their velocities v.

We will apply this postulate to the vacuum polarization due to various charged fields. Expanding the vacuum energy in powers of charge e as the perturbation calculation, we get, to the second order ;

$$U=\frac{n}{(2\pi)^3}\int\{\sqrt{p^2+\mu^2}-eA_0\}d\boldsymbol{P}$$

$$+\frac{ne^2}{3\pi}\left\{\frac{1}{2}w-\mu^2\right\}\frac{\boldsymbol{A}}{4\pi}+\cdots \tag{4}$$

$$w=(p;p')=2\boldsymbol{P}^2+\mu^2=E_P{}^2-(\boldsymbol{P},-\boldsymbol{P}) \quad (4)'^{6)}$$

The second term in the right side of (4) agrees correctly with the self-energy of the photon (1). Accordingly this postulate makes the self-energy of the photon zero.

However, it seems to be difficult to introduce this postulate into the theory of quantized electromagnetic field. In Tomonaga-Schwinger theory, the calculation of the self-energy of a photon in momentum space also gives the result (1).[7] This fact seems to show that the definition of the vacuum in this theory is not in conformity with the above postulate. In our mixture theory[1] or Pauli's regularization method[8] which give the vanishing photon-self-energy, the second term in (4) becomes zero, so that, we can define the vacuum energy as the usual zero-point energy $n\sum_P\sqrt{p^2+\mu^2}$.

From the above investigation we may conclude that our mixture theory of vanishing photon-self-energy has two features : Firstly Weisskopf's correspondence theoretical argument on the zero-point energy is translated in the quantized field theory without difficulty. Secondly, Pauli's regularization method acquires a realistic meaning.

Finally, we apply the above postulate to the problem of the two photons disintegration of a neutral scalar meson. In this case, the interaction term $f\psi^*\beta\psi\varphi$ of the neutral scalar meson φ with the proton ψ replaces the proton mass μ in the equation of motion of the free proton by $\mu+f\phi$. Therefore, let us define the vacuum energy as follows :

$$U=-\frac{2}{(2\pi)^3}\int\Big\{\sqrt{(\boldsymbol{P}-e\boldsymbol{A})+(\mu+f\varphi^2}$$

$$-eA_0\Big\}d\boldsymbol{P} \tag{5}$$

Expanding (5) in powers of charge e and f, we get $\frac{\mu e^2 f}{6\pi^2}\varphi\boldsymbol{A}^2$ as the term of the order e^2f. This term agrees correctly with the non-gauge invariant term which appears in the perturbation calculation of the probability for the two photons disintegration of the neutral scalar meson. Therefore we can get the gauge-invariant result in this case according to the above postulate.

Foregoing correspondence theoretical argument suggests the intimate connection between the problem of gauge-invariancy and the definition of the vacuum.

We wish to acknowledge to Prof. S. Sakata and Dr. T. Inoue for the invaluable discussion and the continued encouragement.

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3) W. Heisenberg : ZS. Phys. **90** (1934), 209.

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On the Spur Calculations in Quantum Mechanics.

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December 8, 1948.

In quantum mechanics, physical quantities are considered to be represented by Hermite operators. We will now seek for an analogy between the Spur and the phase space integral.

The Spur of an operator A is defined by

$$\mathrm{Sp}A=\sum_n(\phi_n,\ A\phi_n)=\int(\phi_k,\ A\phi_k)\ dk, \quad (1)$$

whdre $\{\phi_n\}$ and $\{\phi_k\}$ stand for complete orthonormalized systems such as

$$\sum_n\overline{\phi_n(x)}\ \phi_n(y)=\int\overline{\phi_k(x)}\ \phi_k(y)dk=\delta(x-y) \quad (2)$$

Clearly the definition of Spur is independent of the choice of a special system. For instance, a complete system is given by

$$\phi_k(x)=\frac{1}{\sqrt{2\pi}}\ e^{ik} \quad (3)$$

Using the system (3) we can obtain the Spur of an operator of the type as

$$A=F(Q)G(P), \quad (4)$$

where Q, P stand for the coordinate and momentum operators. In fact,

$$\mathrm{Sp}A=\int dk(\phi_k,\ A\phi_k)=\frac{1}{2\pi}\int dk\int e^{-ikx}$$
$$F(Q)\ G(p)e^{ikx}\cdot dx$$
$$=\frac{1}{2\pi}\iint e^{-ikx}\ F(x)\ G(\hbar k)\ e^{ikx}\ dxdk$$
$$=\frac{1}{2\pi\hbar}\iint F(x)\ G(p)\ dxdp.$$

Thus we get the following formula

$$\mathrm{Sp}(F(Q)G(p))=\iint\frac{dp\,dp}{h}\ F(q)\ G(p). \quad (5)$$

For a system with f degrees of freedom, (5) is easily generalized to

$$\mathrm{Sp}(F(Q_1,\cdots,Q_f)\ G(P_1,\cdots,P_f))=\int\cdots\int$$
$$\frac{dq_1\cdots dq_f\ dp_1\cdots dp_f}{h^f}\ F(q_1\cdots q_f)G(p_1\cdots p_f) \quad (6)$$

This formula is valid for the case when all P's assume continuous eigenvalues, and this enables us to calculate thermodynamical functions of the quantum statistical mechanics by the classical phase space integrals.

1. Application to Statistical Mechanics.

In order to calculate thermodynamical functions, it is enough to know the partition function or the Zustandsumme. In the classical and the quantum statistical mechanics, they are given by

$$Z_{cl}=\int\cdots\int\frac{dq_1\cdots dq_f\ dp_1\cdots dp_f}{h^f}\ e^{-\frac{H}{kT}},$$
$$Z_{qu}=\mathrm{Sp}\left(e^{-\frac{H}{kT}}\right) \quad (7)$$

Now we assume that the system is put into a volume V, then we can assume that the potential U is infinite outside of the given volume V. Moreover, we take a special Hamiltonian as follows:

$$H=K+U=(1/2m)(P_x^2+P_y^2+P_z^2)$$
$$+U(Q_x,\ Q_y,\ Q_z). \quad (8)$$

In the calculation of (7), we use the expansion

$$e^{-\beta H}=e^{-\beta(K+U)}=e^{-\beta U}\ e^{-\beta K}\ (e^{\beta K}e^{\beta U}e^{-\beta(K+U)})$$
$$=e^{-\beta U}\ e^{-\beta K}\ \{1+(1/2)\beta^2$$
$$(KU-UK)+\cdots\}, \quad (9)$$

where $\beta=1/kT$.

If we employ the special form of (8), then readily

$$KU-UK=(\hbar/2mi)\ (\boldsymbol{P}.\ \mathrm{grad}\ U$$
$$+\mathrm{grad}\ U.\boldsymbol{P})$$

$$=(\hbar/2mi)(2\boldsymbol{P}\ \mathrm{grad}\ U-(\hbar/i)\Delta U) \quad (10)$$

follows, where Δ denotes Laplacian. So Z_{qu} is given by

$$Z_{qu}=\mathrm{Sp}(e^{-\beta H})=\mathrm{Sp}[e^{-\beta U}\ e^{-\beta K}\{1+(1/2)\beta^2 (KU-UK)+\cdots\}]$$

$$=\mathrm{Sp}(e^{-\beta U}\ e^{-\beta K})$$

$$+\frac{\hbar\beta^2}{2mi}\ \mathrm{Sp}(e^{-\beta U}\ e^{-\beta K}\ \boldsymbol{P}\ \mathrm{grad}\ U)$$

$$+\frac{\beta^2\hbar^2}{4m}\ \mathrm{Sp}\ (e^{-\beta U}\ e^{-\beta T}\ \ \Delta U)+\cdots\cdots \quad (11)$$

While

$$\mathrm{Sp}(e^{-\beta U}\ e^{-\beta K})$$

$$=\int_V\cdots\int_{-\infty}^{\infty}\frac{dq_1dq_2dq_3\,dp_1dp_2dp_3}{h^3}\ e^{-\beta H}$$

$$=Z_{cl}\ ,$$

and the rest terms are

$$\mathrm{Sp}(e^{-\beta U}\ e^{-\beta K}\ \boldsymbol{P}\ \mathrm{grad}\ U)=\mathrm{Sp}\ (\mathrm{grad}\ U.\ e^{-\beta U}$$

$$e^{-\beta K}\ \boldsymbol{P})=h^{-3}\int_V dq_1dq_2dq_3\ \mathrm{grad}\ U.e^{-\beta U}$$

$$\int\boldsymbol{P}e^{-\beta K}\,dp_1dp_2dp_3=0.$$

Now we will put for any quantity S that

$$<\mathrm{S}>_{cl.}=\frac{1}{Z_{cl}}\int\cdots\int^{\infty}\frac{dq_1\cdots dq_f dp_1\cdots dp_f}{h^f}\ \mathrm{S}e^{-\beta H} \quad (12)$$

then we obtain

$$Z_{qu}=Z_{cl}\ \left(1+\frac{\hbar^2}{4mk^2T^2}<\Delta U>_{cl}+\cdots\right) \quad (13)$$

This is a power series of $\hbar^2$, explicitly representing the relation between classical and quantum statistical mechanics.

2. Expansion Theorem.

Another application of the Spur calculaion is the expansion theorem of an operator. Consider such an operator in Hilbert space $\mathfrak{H}$ as

$$\mathrm{Sp}(\mathrm{S}^*\mathrm{S})<+\infty, \quad (14)$$

where S* is the adjoint operator of S.

We will call the set of all operators satisfying (14) as $\mathfrak{S}$. The property of $\mathfrak{S}$ is as follows :

(1) $\mathfrak{S}$ is a linear space. If $\mathfrak{S}\in\mathrm{S}_1,\ \mathrm{S}_2$, then $\mathfrak{S}\in\alpha\mathrm{S}_1+\beta\mathrm{S}_2$, where α, β are complex numbers.

(2) If $\mathfrak{S}\in\mathrm{S}_1,\mathrm{S}_2$, then $\mathrm{Sp}(\mathrm{S}_1{}^*\ \mathrm{S}_2)<+\infty$. And it is easily proved that $\mathfrak{S}$ forms a Hilbert space.

In this case the product, or more precisely the inner product of δ_1 and δ_2 is defined by

$$(\mathrm{S}_1,\ \mathrm{S}_2)=\mathrm{Sp}\ (\mathrm{S}_1{}^*\mathrm{S}_2) \quad (15)$$

We must notice that the element of $\mathfrak{S}$ is an operator in the ordinary Hilbert space $\mathfrak{H}$.

Let us denote the operator in $\mathfrak{S}$ by the German letters. Clearly the following operator $\mathfrak{L}$ is linear,

$$\mathfrak{L}\mathrm{S}=L\mathrm{S}-\mathrm{S}L, \quad (16)$$

where L is an operator in $\mathfrak{H}$ and S an element of $\mathfrak{S}$.

If L is Hermitic in $\mathfrak{H}$, then So is $\mathfrak{L}$ in $\mathfrak{S}$.

We define two operators $\mathfrak{P}$, $\mathfrak{Q}$ by P, Q as

$$\mathfrak{P}\mathrm{S}=P\mathrm{S}-\mathrm{S}P,\quad \mathfrak{Q}\mathrm{S}=Q\mathrm{S}-\mathrm{S}Q. \quad (17)$$

But by Jacobi's identity

$$[P[Q,\ \mathrm{S}]]=[Q[P,\ Q]\mathrm{S}]$$
$$=[Q[P,\mathrm{S}]]+[\hbar/i,\mathrm{S}]$$
$$=[Q[P,]\mathrm{S}],$$

which signifies for $\mathfrak{P}$ and $\mathfrak{Q}$ that

$$\mathfrak{P}\ \mathfrak{Q}=\mathfrak{Q}\ \mathfrak{P}. \quad (18)$$

By this commutativity we can discuss about the following simultaneous eigenvalue problem, i.e.

$$\begin{cases}\mathfrak{P}\ \mathrm{S}=p\mathrm{S}\\ \mathfrak{Q}\mathrm{S}=-q\mathrm{S}\end{cases} \quad (19)$$

Then the eigenvalues are $-\infty<p<+\infty, -\infty<q<+\infty$ and the eigenfunctions for the given p, q is expressed by

$$S=\sigma(p,q)=\exp\left(\frac{i}{\hbar}pQ\right)\exp\left(\frac{i}{\hbar}qP\right) \quad (20)$$

and the solution is unique.

By the theory of Hilbert space, any element S of $\mathfrak{S}$ can be expanded as

$$S=\iint\frac{dp\,dq}{h}\,c(p,q)\,\sigma\,(p,q), \qquad (21)$$

because of the completeness of the eigenfunctions of Hermite operators.

Now remember the formula (5), then using Fourier's integral theorem and Weyl's equation, $c(p,\ q)$ is given by

$$c(p,\ q)=\mathrm{Sp}(\sigma(p,\ q)S^*)=(\sigma(p,\ q),S). \qquad (22)$$

Consequently we can write simply

$$S=\iint\frac{dp\,dq}{h}\,(\sigma(p,\ q),\ S)\ \sigma\ (p,q). \qquad (23)$$

Now multipiying S^* from the left side of (23), and taking the Spur, we get the following formula analogous to Parseval's:

$$\mathrm{Sp}(S^*S)=\iint\frac{dp.dq}{h}\,|(\sigma(p,q),\ S)|^2. \qquad (24)$$

This expansion theorem shows the reciprocity between P and Q, and can be applied to the non-localizable field theory.

Quantum Theory of Generalized Local Fields

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December 8, 1949.

Recently, Koba and the present author[1)2)] proposed the new method which can describe more general interactions than those in the current one. The essential point of our generalization was as follows: One considers the system having the interaction Lagrange density containing derivatives of the field variables of any order and being a Loreniz-invariant analytic function of these quantities. Then, one assumes that the quantum mechanical system corresponding to the above stated classical one is described by the following Tomonaga-Schwinger equation:

$$\left\{H[x:\sigma]+\frac{\hbar}{i}\frac{\delta}{\delta\sigma(x)}\right\}\Psi[\sigma]=0. \quad (1)$$

where the field variables contained in $H[x;\sigma]$ satisfy the same fourdimensional commutation relations as the free field ones, and the generalized interaction Hamilton density $H[x;\sigma]$ of our system is derived from the interaction Lagrange density $L(x)$ such that (i) it satisfies the integrability condition

$$\begin{aligned}&[H[x;\sigma(x)]+\frac{\hbar}{i}\frac{\delta}{\delta\sigma(x')},\\ &H[x';\sigma(x')]+\frac{\hbar}{i}\frac{\delta}{\delta\sigma(x')}]=0,\end{aligned} \qquad (2)$$

where X and X' are space-like to each other and (ii) the equation (1) reduces to the ordinary one when the canonical formulation is possible. It is to be noticed that the operator $H[x;\sigma]$ obtained here contains, in general, not only the unit normal of the hyper surface σ but its derivatives, so that it depends not only on the inclination of the hyper surface σ at the point X but on curvature etc. of σ at this point.

Now in this letter the present author answers the questions how the energy-momentum conservation law will be formulated in this general case. The total energy-momentum quantities of our system are defined as follows:

$$\begin{aligned}P_\nu[x;\sigma(x)]=&\int_\sigma T^\circ_{\mu\nu}(x')d\sigma_\mu(x')\\ &+\delta_{\mu\nu}\int_\sigma H[x';\sigma(x')]d\sigma_\mu(x'),\end{aligned} \qquad (3)$$

where $T^\circ_{\mu\nu}$ means the energy-momentum tensor of the free fields and $H[x;\sigma]$ is the interaction Hamilton density. It should be clear that the energy-momentum conservation law of our system, in the interaction representation, it is to be written

$$\frac{\Delta P_\nu[x;\sigma]}{\Delta\sigma(x)}=\frac{\delta P_\nu[x;\sigma]}{\delta\sigma(x)}$$
$$+\frac{i}{\hbar}[H[x;\sigma],\ P_\nu[x;\sigma]]=0. \tag{4}$$

The equation (4) follows at once by noticing that the relation

$$\frac{\Delta P_\nu[x;\sigma]}{\Delta\sigma(x)}=\frac{\hbar}{i}\int_{o}[H[x;\sigma(x)]+\frac{\hbar}{i}\frac{\delta}{\delta\sigma(x)},$$
$$H[X';\sigma(x')]+\frac{\hbar}{i}\frac{\delta}{\delta\sigma(x')}]\,d\sigma_\nu(x') \tag{5}$$

holds. According to (2) the right-hand side of (5) equals to zero, so that the conservation law (4) is correct. It will be remarkable that the integrability condition of the Tomonaga-Schwinger equation is perfectly equivalent to that of the energy-momentum law. Moreover, it is to be noticed that there exists in our general formulation a possibility of disposing of some part of the socalled interaction type divergencies which cannot be removed consistently by the present Tomonaga-Schwinger theory.

Detailed accounts and especially the last mentioned facts will be clarified in the later issue of this journal.

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Excess Electrons in Lower Atmosphere and γ-decay of Neutral Mesons

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December 9, 1949.

It is well known that electrons in the lower atmosphere can not be explained by only the knock-on and the decay of μ mesons[1]. Rossi ascribed such electrons to N-radiations, which are considered to be produced by the nucleon component and to responce to high energy showers associated with penetrating showers[2]. In order to inquire the substance of N-radiation, we proposed the existence of β-meson; which is slightly lighter than π-meson and responces to nuclear force and β-decay.[3] Although the existence of such a meson is disproved by Berkeley experiment, its role in cosmic rays may be remained if the substance is adequately replaced by other one. This replaced substance may be the neutral π-meson which rapidly disintegrates into two photons. Indeed we can explain the intensity of soft rays in the upper atmosphere by means of neutral mesons[3)5)]. We will attempt to ascribe the excess electrons in the lower atmosphere to neutral mesons, since their role is almost certainly established by various arguments.

The number of neutral mesons produced per g cm^{-2} air at the atmospheric depth x with energy between ε and $\varepsilon+d\varepsilon$ is given by

$$f(\varepsilon,\ x)\ d\varepsilon=\gamma A\ \exp\ (-x/125)\ \varepsilon^{-\gamma-1}d\varepsilon, \tag{1}$$

where $A=4.6\times10^{-6}\ \mathrm{sec}^{-1}\mathrm{cm}^{-2}$, measuring the energy in 10 Bev as before[5]. The number of electrons in equilibrium with parent nucleons which may be a good approximation in lower altitudes under consideration, is obtained by the method of electron track. The length of electron track initiated by the energy ε is

$$Z(\varepsilon)=(\varepsilon/\varepsilon_j)X_0,\ \varepsilon_j=86\ \mathrm{Mev},$$
$$X_0=36\ \mathrm{gcm}^{-2}. \tag{2}$$

The number of electrons is given by

$$N(x)=\int_{\varepsilon_j}^{\infty}f(\varepsilon,\ x)\ Z(\varepsilon)d\varepsilon$$
$$=\frac{\gamma}{\gamma-1}\ Ae^{-x/125}\ \varepsilon_j{}^{-\gamma}. \tag{3}$$

The result is represented in Fig. 1 by a dashed curve D for the altitudes lower than 600 g cm^{-2}.

The experimental result to be compared is taken from Rossi's articls[2], represented by curve A. The contribution from the knock-on and decay electrons of μ-mesons is determined by referring to the analysis of Bernardini et al[1], but by accounting for that the decay of a μ-meson is modified as to disintegrate into one electron and two neutrinos, as indicated by curve B. A−B=U represents the contribution from N-radiations. The agreement between curves C and D, besides a slight inevitable discrepancy in the absolute values, seems to mean that N-radiation may be identified with the decendant of the neutral meson. Thus the γ-decay of neutral mesons is. in our opinion, almost thoroughly established in cosmic ray phenomena.

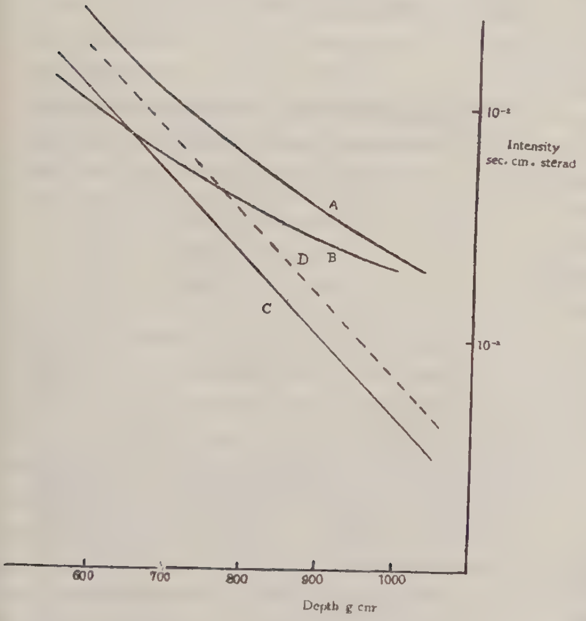

Fig. 1. The intensity of electrons in the lower atmosphere.
A : whole electrons.
B : electrons from μ-mesons.
C : electrons from N-radiations. A-B.
D: electrons from neutral mesons.

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2) B. Rossi: Rev. Mod. Phys., **20** (1948), 537.
3) M. Sasaki, S. Nakamura and S. Hayakawa: Prog. Theor. Phys., **4** (1948), 454.
4) S. Hayakawa and J. Nishimura: Prog. Theor. Phys., in press.
5) V. Fujimoto and S. Hayakawa. Prog. Theor. Phys. in press.

On the Concept of the Nuclear Potential.

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From the very beginning of the meson theory proposed by Yukawa, the traditional concept of the nuclear potential does not seem to have experienced any progress. The nuclear potential has been taken as an adiabatic or a static potential between nucleons, quite analogous to the Coulomb interaction between charged particles. The latter can, however, be derived from the fundamental equation of field theory by a rigorous canonical transformations containing no approximation, and its validity has been confirmed by a great many experimental facts. On the contrary, the usual nuclear potential is calculated only up to the second order with respect to the coupling constant of the nucleon and the meson. It's validity seems, at first sight, to be restricted by the applicability of the perturbation method, and the situation is serious because the coupling constant is very large and the higher order terms become larger for higher energies for the types of interaction including derivatives of the meson field variables. In fact Bethe⁽¹⁾, Watson and Lepore⁽²⁾ showed that the fourth order terms have an important contribution to the nucleonic collision in the energy range of about 100 Mev.

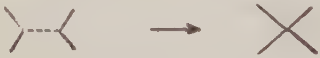
In order to determine the types of mesons and of interactions, two groups of data must be used. One is the collision between nucleons and mesons for which the straight-forward perturbation calculations may be sufficient if all necessary reactions, both reactive and resistive, are taken into account. The concept of nuclear potential becomes essential in the other case of the deuteron where the deformation of the wave function is large and the validity of Born approximation is lost.

In this short note we want to remark a

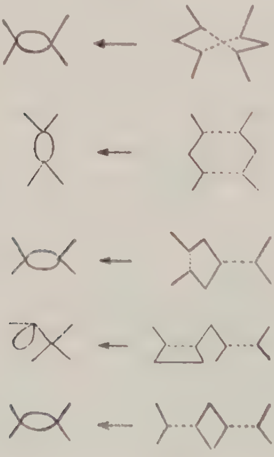
simple fact that only the second order Møller interaction must be taken as the rigorous nuclear potential for any strength of the coupling and the effect observed by Watson and Lepore is nothing but the effect of deformation of the wave function due to the Møller interaction. This will easily be seen in the following diagram. It may also be inferred that on solving the deuteron states we need use only the second order Møller interaction and solve the relativistic differential equation rigorously. The relativistic treatment is needed in order to take account of the transitions through negative energy states which become important especially for the pseudoscalar mesons. As pointed out by Arkaki,(3) Toyoda and Nambu,(4) the transformation of the Møller interaction into the space-time coordinates of the nucleons is ambiguous, but it is necessary to note that all the terms, which does not vanish identically for virtual transitions of the nucleons, must be retained.

Details of the calculation will be published in the near future.

Feynman-Diagram of the 2nd order Møller Interaction



All the 4th order processes are represented as the combinations of the 2nd order processes :



Each process of above diagrams corresponds to *Sa*, *Sb*, *Sc*, *Sd*, *Se* of Watson-Lepore's processes, respectively.

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3) G. Araki, Private Communication to Mr. Toyoda. Prog. Theor. Phys. **4** (1949), 193.
4) Toyoda, Lecture before the symposium on the theory of elementary particles held on Dec. 8 and 9, 1949.
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On the Interaction between Nuclear-Vibrational and Electronic States.

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December 19, 1949.

By group-theoretical considerations could be proved the general theorem[1] : all degenerate electronic states, without special twofold one, of non-linear nuclear configurations in the polyatomic molecular systems are unstable whether the degeneracy is due to electronic orbitals or to spin. Only the linear configurations in degenerate electronic states, however, can be stable with respect to all displacements of the nuclei. Since orbital two-fold degeneracies are essentially due to the axial symmetries of the nuclear configurations, they may split by breaking their linearity, e.g. the nuclear displacements of the normal mode of a deformation vibration, and it is expected that the magnitude of the splitting of their levels is as same order as the potential energy of the nuclear vibration. Under these conditions, therefore, we have to reconsider the ordinary separation into electronic and nuclear motions[2].

Accordingly we restrict our problem to the interaction between electronic and deformation-vibrational states in question of the linear XY_2-type molecules. It is also possible to extend our considerations to all other linear systems. The wave function of such a system can be given by

$$\Psi\ (\vec{q}_e,\ \vec{q}_n)=\chi^+(\vec{q}_n)\overset{+}{\psi}_n(\vec{q}_e)+\chi^-(\vec{q}_n)\ \overset{-}{\psi}_n(\vec{q}_e),$$

where $\overset{+}{\psi}_n(\vec{q}_e)$ and $\overset{-}{\psi}_n(\vec{q}_e)$ belonging to the eigenvalues $E_n{}^e\pm\sigma\ (r)$ are symmetric and antisymmetric electronic functions for the reflection by the plane containing the principal axis, respectively, and $\sigma(r)$ principally determined by perturbation calculations is the half value of the electronic separation which gives the magnitude of the interaction between electronic and nuclear motions. So we can obtain the following Schroedinger's equation for two-dimensional vector X, whose components are the nuclear functions $\chi^+(r,\ \varphi)$ and $\chi^+(r,\ \varphi)$ mentioned above:

$$\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\left(\frac{\partial}{\partial\varphi}-\Lambda P\right)^2\right]$$
$$X(r,\varphi)+\frac{2\mu}{\hbar^2}[E-E_n{}^e+Q.\sigma(r)]X(r,\varphi)=0,$$

where $P=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$, $Q=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$,

and Λ (unit by $\hbar$) is the electrontic angular momentum about the principal axis, μ being the reduced mass of this vibration. Then the zeroth order harmonic solution is

$$\overset{(0)}{X}_{\lambda Vm}=\begin{cases}\overset{(0)+}{\chi}_{\lambda Vm}=(1/\sqrt{4\pi})e^{i(m\pm\Lambda)\varphi}\\ \overset{(0)-}{\chi}_{\lambda Vm}=\pm(1/\sqrt{4\pi})^{i(m\pm\Lambda)\varphi}\end{cases}$$
$$\left.\begin{matrix}R_{V\,|m|}(r)\\ R_{V\,|m|}(r)\end{matrix}\right\}\ \left(\pm\text{for }\lambda=\begin{Bmatrix}1\\2\end{Bmatrix}\ \right),$$

where V is the vibrational quantum number, $|m|=V-2\mathrm{i}$ ($i=0$ or a positive integer) means that of nuclear angular momentum, and $R_{V\,|m|}(r)$ is the solution of the radial part including the associated Laguerre function. Thereupon we can define $M=m\pm\Lambda$, the quantum number of the total angular momentum about the principal axis.

If we assume the Π-term as the electronic ground state from the fact mentioned above, it becomes indispensable to consider the Δ-term 3). It can be easily shown that the perturbed splitting of the Δ-term caused by a electric dipole due to the nuclear displacement is contributed by 4th order perturbation, while the quadrupole effect of this system contributes to 2nd order, and then the splitting due to both effects is proportional to the fourth power of the displacement: $\sigma(r)=\eta r^4$. In such a case we have to consider the anharmonicity correction also in the vibrational potential. Therefore the total-symmetric and anti-symmetric modes of the nuclear displacements will be introduced, but they cannot contribute to destroy the linearity. Consequently, we can put $E^e{}_n(r)=\alpha r^2+\beta r^4$, where β is dfferent from the pure anharmonicity constant, and the energy correction of the quartet term is readily obtained.

Such being the case we can construct the interaction energy matrix for the perturbation term $S=-Q\eta r^4$:

$$(\lambda VM|S|\ \lambda' V'M')=-(\eta/4\pi)\int_0^\infty$$
$$\int_0^{2\pi}\overset{(0)}{X}{}^*_{\lambda Vm}\ Qr^4\ \ \overset{(0)}{X}_{\lambda' V' M'}\ rdr\ d\varphi,$$

and in general $(\lambda VM|S|\lambda V'M')=0$ $(\lambda=1,2)$, where the selection rules are given by $M'=M$ $(m'=m\pm4)$, and $V'=V$, $V\pm2$, $V\pm4$. When performing to S the transformation to principal axes by a suitable unitary matrix, we can find the perturbation energy in addition to the vibrational proper term. The first order corrections which cause the splitting are given by

$$\overset{(1)\pm}{E}_{M+2i-2\pm M}=(1,M+2i+2,\pm M|S|2,M$$
$$+2i-2,\pm M)$$
$$=\pm3\varepsilon h\nu_1\sqrt{i(i-1)(M+i)(M+i-1)},$$

where $\varepsilon=\sigma/E^e{}_n$, and no splitting take place in the case of $i=0$ or 1. The second order shift energies were also obtained as follows:

$$E^{(2)\pm}_{M+2i-2\pm M}$$
$$= -\varepsilon^2 h\nu[(M+2i)(M+2i-1)(M+2i-2)$$
$$-\frac{1}{8}(M+2i-1)\,(2M^2+2Mi-M+2i^2-2i)].$$

From above theoretical considerations it will become possible to criticize the interaction between nuclear and electronic states through the analysis of the vibrational structure associated with the electronic *Π-Δ* transition.

1) H. A. Jahn and E. Teller, Proc. Roy. Soc. London, A **161** (1937), 220.
2) M. Born and J. R. Oppenheimer, Ann. d. Phys. **84** (1927), 457.
3) H. Sponer and E. Teller, Rev. Mod. Phys. **13** (1941), 90 The treatment of *Π*-term is similar to R. Renner [ZS. f. Phys. **92** (1934), 172]

On the Neutron-Proton Scattering

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December 26, 1949.

In the analysis of the scattering experiments, much laborious computations are required to get the phase shifts in any approximation. In order to avoid this trouble we made use of the Differential Analyzer, by means of which we solved the Schrödinger equations exactly, and calculated the total cross section and angular distribution of the neutron-proton scattering both for slow and fast neutrons.

We took Yukawa potential $A \exp(-\kappa r)/r$. The force range was determined from the π-meson mass $m_V = 284 m_e$, and the force constant A (or $\beta = -MA/\hbar^2\kappa$ as convenient for calculation)* from the binding energy of deuteron $E = -2.19$ Mev. Tensor force was not taken into account, because our machine has only 4 integrators which are inadequat toe solve the simultaneous equations. The results are shown in Table I. For reference we studied how β changes with the meson mass m_V in the range between $200 m_e$ and $400 m_e$.

Table I. Relation between range and depth

m_V	β	$a(\times 10^{-13}\,cm)$	$V=-A\kappa$ (Mev)
200	−2.86	1.96	31.5
240	−2.66	1.60	42.1
260	−2.52	1.48	47.1
284	−2.45	1.35	54.5
300	−2.40	1.28	59.6
320	−2.35	1.20	66.6
340	−2.31	1.13	73.7
360	−2.27	1.07	81.2
380	−2.25	1.01	90.3
400	−2.22	0.96	98.2

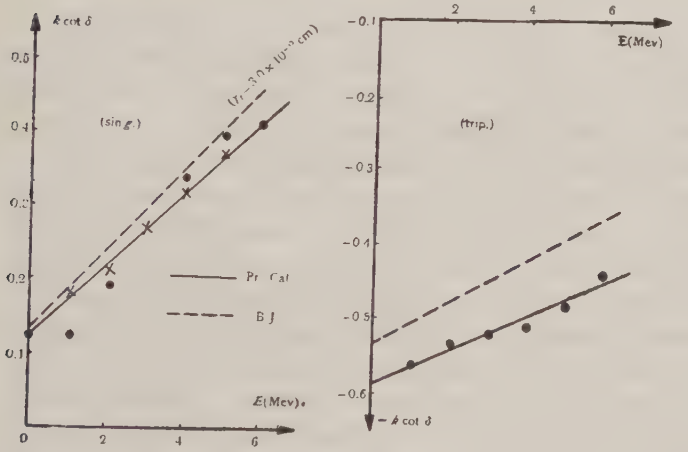
For the singlet state, we assumed attractive force with the same force range as in the case of the triplet state and determined β_s from the total cross section for thermal neutron 20.3×10^{-24} cm² [1]. In varying the potential depth, the cross section becomes infinite at $\beta_s = -1.7$, and at its both sides, two β_s's (-1.57 and -1.83) fit the experimental values. For $\beta_s = -1.83$ there should exist a real bound state of the deuteron, but none for $\beta_s = -1.57$. But we abandoned $\beta_s = -1.83$, because it gives the scattering amplitude quite disagreed with the other experimental data.

The calculated cross sections for the neutron with energy 1∼6 Mev are compared with the experimental ones of Williams *et al*[1] in Table II.

Table II. The total cross section.

E_{lab} (Mev)	σ (in barns) cal.	exp.
1	4.01	4.15 ± 0.15
2	2.60	2.96 ± 0.07
3	2.02	2.33 ± 0.13
4	1.63	1.85 ± 0.09
5	1.42	1.63 ± 0.05
6	1.29	1.32 ± 0.12

Ingeneously Schwinger[4)] found $k.\cot\delta$ could be expressed as a linear function of the incident energy in the case of slow neutrons. Blatt and Jackson[5)] performed the practical calculations with purpose to verify the Schwinger's shape independent theory. According to them the force range should be 1.2×10^{-13}cm. The comparison of our results with theirs is shown in Fig. 1.



In the next, we calculated the total cross section at 90 Mev with the above-determined potentials. The phase shifts were computed by means of the analyzer and compared with those by the Born approximation, in Table III.

Table III. Phase shifts at 90 Mev

	Ezact (trip.)	(sing.)	Born (trip.)	(sing.)
δ_0	1.107	.608	1.902	1.219
δ_1	.404	.147	.649	.416
δ_2	.217	.087	.264	.169
δ_3	.038	.012	.076	.073
δ_4	.008	.004	.032	.030
δ_5	.006	.006	—	—
	$\sigma_{total}=0.134\times10^{-24}$ cm^2		0.242×10^{-24}cm^2	

Would it be possible to reduce the total cross section to the experimental value by shortening the force range to Blatt-Jackson's value? The behaviour of the triplet phase shifts in the range of the meson mass 260∼340m_e is shown in Table IV. The phase shifts are insensitive to the force range in this region.

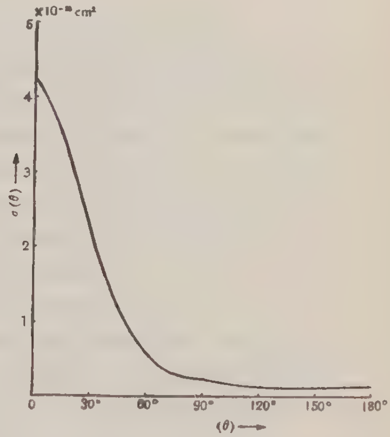


Fig. 2. Differential cross section for neutron-proton scattering at 90 Mev

Table IV. Phase shifts at 90 Mev for various ranges

m_U	$l=0$	$l=0$	$l=2$
260	1.10	.426	—
284	1.11	.404	.217
300	1.14	.387	.191
320	1.14	.346	.124
340	1.15		.116

Finally we calculated the angular distribution at 90 Mev using the phase shifts in Table III. The results are shown in Fig. 2.

We wish to express our sincere thanks to Prof. Yamanouchi for his kind guidance.

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2) R. B. Sutton *et al.*, Phys. Rev. **72** (1947), 1147.
C. G. Shull *et al*, Phys. Rev. **73** (1948), 842.
3) J. H. Williams *et al.*, Phys. Rev. **70** (1946), 583.
4) J. Schwinger, Phys. Rev. **72** (1947), 742A.
5) J. M. Blatt and Jackson, Phys. Rev. **76** (1949), 18.

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On the Decay of Heavy Mesons III.* ** ***

— *Case of the Unsymmetrical Interaction.* —

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1. Introduction

In the previous works,[1] the decay schemes of heavy mesons called "Case of the Symmetrical Interaction" were treated, where the following processes were considered

$$\tau^+ \rightarrow \pi^+ + \pi^0,\ \tau^+ \rightarrow \tau^0 + \pi^+,\ \tau^0 \rightarrow \pi^0 + \gamma.$$

In the attempts to explain the decay of τ meson through the intermediary of nucleon field, the γ-instability of a charged τ meson should not be omitted. In this paper, this problem is treated in detail using Tomonaga-Schwinger's formalism[2]. As a photon interacts with a proton but does not with a neutron, the matrix elements for this process become unsymmetrical with regard to nucleons. This is the reason why we call this problem "Case of the Unsymmetrical Interaction". Moreover, owing to the existence of a photon there must be added the processes where a charged meson interacts with a photon.

Besides the problem of searching for the decay schemes of τ meson consistent with experiments, there exist theoretically interesting problems concerning with the divergence, gauge invariance and equivalence theorem of meson-nucleon-photon coupling. In this connection the examinations of the applicability of Pauli's regulator[4] were made.

After the analysis of all cases, it turns out that the following combinations of τ and π meson are forbidden.

(i) The cases when both τ and π meson have spin 0.

(ii) The cases when τ or π meson is the scalar meson with vector coupling. In another cases the life times of the decay of a charged τ meson into a charged

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*** We wish to thank Prof. Yukawa, who has kindly teached us the importance of the decay of a heavy meson into a lighter meson and a photon instead of the decay of a π meson into a μ meson and a photon.

π meson and a photon turn out to be shorter than 10^{-11} sec. even if we drop the non-gauge invariant terms by regulator. So the life-times of the decay processes of charged τ mesons discussed here and in reference (1) will be too short to be compatible with the experiments. It is quite natural to suppose that the above mentioned processes should be forbidden in order to explain the experiments by Powell and others.[5]

Putting together the selection rules obtained for the processes discussed here and in reference (1), these are the cases when

(i) both τ and π meson are pseudoscalar mesons

(ii) $\tau(\pi)$ meson is pseudoscalar and $\pi(\tau)$ is scalar.

According to the discussions in reference (1), if we choose suitably for neutral π meson symmetrical or neutral theory, the following processes can also be prohibited

(iii) both τ and π meson are scalar mesons

(iv) $\tau(\pi)$ meson is scalar meson with vector coupling.

These discussions will be made in our paper IV.

2. Interaction representation and canonical transformation

The generalized Schrödinger equation for the system consisting of nucleons, photons, τ and π mesons becomes as follows;

$$i\hbar c\frac{\delta\Psi[\sigma]}{\delta\sigma(x)}=\{H_1(x;\sigma)+H_2(x;\sigma)\}\Psi[\sigma] \tag{1}$$

$$\begin{cases} H_1(x;\sigma)=H_G(\tau,n)+H_{Ge}(\tau,n,\gamma;\sigma) \\ H_2(x;\sigma)=H_e(n,\gamma)+H_e(\pi,\gamma)+H_e(\tau,\gamma) \\ \qquad\qquad +H_g(\pi,n)+H_{ge}(\pi,n,\gamma;\sigma) \end{cases}$$

where

$$H_e(n,\gamma)=-ie\bar{\psi}\tau_p\gamma_\mu\psi A_\mu,\quad \bar{\psi}=\psi^*\gamma_4. \tag{2}$$

Interaction Hamiltonians for photon-meson and nucleon-meson system are given by Tomonaga and his collaborators[6]. Furthermore, we construct nucleon-vector (pseudo-vector) meson interaction using Proca formalism for the meson field and nucleon-meson-photon interactions which are in agreement with the results obtained by G.J. Yevick[7] independently. For the scalar (pseudoscalar) τ meson, we get

$$H_G(\tau,n)=G_1U^*W+\frac{G_2}{\varkappa_\tau}\frac{\partial U^*}{\partial x_\alpha}P_\alpha+\frac{G_2^2}{\varkappa_\tau^2}2\pi P_\alpha^*P_\beta N_\alpha N_\beta+C.C., \tag{3}$$

$$H_e(\tau,\gamma)=\frac{1}{4\pi}\left(\frac{ie}{\hbar c}\right)A_\alpha\left(U^*\frac{\partial U}{\partial x_\alpha}-\frac{\partial U^*}{\partial x_\alpha}U\right)-\frac{1}{4\pi}\left(\frac{ie}{\hbar c}\right)^2U^*U\times$$
$$\times\{A_\alpha^2+(N_\alpha A_\alpha)^2\}, \tag{4}$$

$$H_{Ge}(\tau, n, \gamma)=\frac{G_2}{\varkappa_\tau}\left(\frac{ie}{\hbar c}\right)A_\alpha(U^* P_\alpha-P_\alpha^* U)+\frac{G_2}{\varkappa_\tau}\left(\frac{ie}{\hbar c}\right)(A_\alpha N_\alpha P_\beta N_\beta U^*$$
$$-A_\alpha N_\alpha P_\beta^* N_\beta U), \tag{5}$$

$$W=\bar{\psi}(i\gamma_5)\tau_{NP}\psi, \qquad P_\alpha=i\bar{\psi}(\gamma_5)\gamma_\alpha\tau_{NP}\psi,$$
$$\varkappa_\tau=\frac{m_\tau c}{\hbar}.$$

For the vector (pseudo-vector) τ meson we get

$$H_G(\tau, n)=-G_1 U_\alpha^* P_\alpha+\frac{G_2}{2\varkappa_\tau} U_{\alpha\beta}^* S_{\alpha\beta}+\frac{4\pi}{\varkappa_\tau^2} G_2^2 \frac{S_{\alpha\beta} S_{\alpha\beta}^*}{4}+\frac{4\pi}{\varkappa_\tau^2}\left\{\frac{G_1^2}{2} M_\alpha^* M_\beta N_\beta N_\alpha\right.$$
$$\left.+\frac{G_2^2}{2} S_{\alpha\mu}^* S_{\alpha\nu} N_\mu N_\nu\right\}+C.C., \tag{6}$$

$$H_e(\tau, \gamma)=\frac{1}{8\pi}\left[\left(\frac{ie}{\hbar c}\right)\{(A_\alpha U_\beta^*-A_\beta U_\alpha^*)U_{\alpha\beta}-U_{\alpha\beta}^*(A_\alpha U_\beta-A_\beta U_\alpha)\}\right.$$
$$-\left(\frac{ie}{\hbar c}\right)^2\{(A_\alpha U_\beta^*-A_\beta U_\alpha^*)(A_\alpha U_\beta-A_\beta U_\alpha)+2(A_\alpha U_\beta^*-A_\beta U_\alpha^*)$$
$$\times(A_\alpha U_\mu-A_\mu U_\alpha) N_\beta N_\mu$$
$$\left.+\frac{2}{\varkappa_\tau^2} A_\alpha A_\beta U_{\alpha\mu}^* U_{\beta\nu} N_\mu N_\nu\}\right]. \tag{7}$$

$$H_{Ge}(\tau, n, \gamma)=\frac{-G_1}{\varkappa_\tau^2}\left(\frac{ie}{\hbar c}\right)\{A_\alpha \chi_{\alpha\beta} P_\gamma^* N_\beta N_\gamma-A_\alpha \chi_{\alpha\beta}^* P_\gamma N_\beta N_\gamma\}$$
$$+\frac{G_2}{2\varkappa_\tau}\left(\frac{ie}{\hbar c}\right)\{(A_\alpha U_\beta^*-A_\beta U_\alpha^*) S_{\alpha\beta}-(A_\alpha U_\beta-A_\beta U_\alpha) S_{\alpha\beta}^*$$
$$+2(A_\alpha U_\beta^*-A_\beta U_\alpha^*) S_{\alpha\gamma} N_\gamma N_\beta-2(A_\alpha U_\beta-A_\beta U_\alpha) S_{\alpha\gamma}^* N_\gamma N_\beta\}, \tag{8}$$

$$S_{\alpha\beta}=-i\bar{\psi}(\gamma_5)\gamma_\alpha\gamma_\beta\tau_{NP}\psi, \quad U_{\alpha\beta}=\frac{\partial U_\beta}{\partial x_\alpha}-\frac{\partial U_\alpha}{\partial x_\beta}.$$

where N_α is the component of the normal to the surface σ and the other quantities express the usual meaning. For the π meson, we have only to replace $\varkappa_\tau$ by $\varkappa_\pi$ and G by g in the above expressions. The commutation relations between these field variables become as follows:

$$[U(x), U^*(x')]=4\pi i\hbar c D_\tau(x-x'), \tag{9}$$

$$[U_\alpha(x), U_\beta^*(x')]=4\pi \hbar c\left(\delta_{\alpha\beta}-\frac{1}{\varkappa_\tau^2} \frac{\partial^2}{\partial x_\alpha \partial x_\beta}\right) D_\tau(x-x'), \tag{10}$$

$$\{\psi_\alpha(x), \bar{\psi}_\beta(x')\}=-iS_{\alpha\beta}(x-x')\tau_N-iS_{\alpha\beta}(x-x')\tau_P, \tag{11}$$

$$S_{\alpha\beta}(x-x')=\left(\gamma_\mu \frac{\partial}{\partial x_\mu}-\varkappa\right)\Delta(x-x'),\quad \varkappa=\frac{Mc}{\hbar},\quad M=\text{nucleon mass},$$

other commutators$=0$.

As our process occurs through the coupling Gge, it is convenient to make the following canonical transformation

$$\Psi[\sigma]=U[\sigma]\,\Psi'[\sigma] \tag{12}$$

where $U[\sigma]$ satisfies the relation

$$i\hbar c\,\frac{\delta U[\sigma]}{\delta\sigma(x)}=H_2(x\,;\,\sigma)\,U[\sigma]. \tag{13}$$

By this transformation we get

$$i\hbar c\,\frac{\delta \Psi'[\sigma]}{\delta\sigma(x)}=U^{-1}[\sigma]\,H_1(x\,;\,\sigma)\,U[\sigma]\,\Psi'[\sigma]. \tag{14}$$

Then in the order Gge our generalized Schrödinger equation reads

$$i\hbar c\,\frac{\delta \Psi'[\sigma]}{\delta\sigma(x)}=(P+Q+R)\,\Psi'[\sigma],$$

$$P=A+B,\quad A=-\frac{i}{\hbar c}\int_{-\infty}^{\sigma}d\omega'\,[H_{Ge}(\tau,n,\gamma\,;\,\sigma),\ H_g'(\pi,n)],$$

$$B=-\frac{1}{(\hbar c)^2}\int_{-\infty}^{\sigma}d\omega'\int_{-\infty}^{\sigma'}d\omega''\,\{[H_G(\tau,n),\ H_e'(\tau,\gamma)],\ H_g''(\pi,n)]$$
$$+\{[H_G(\tau,n),\ H_g'(\pi,n)],\ H_e''(\tau,\gamma)]\},$$

$$Q=C+D,\quad C=-\frac{i}{\hbar c}\int_{-\infty}^{\sigma}d\omega'\,[H_G(\tau,n),\ H_{g3}'(\pi,n,\gamma\,;\,\sigma)],$$

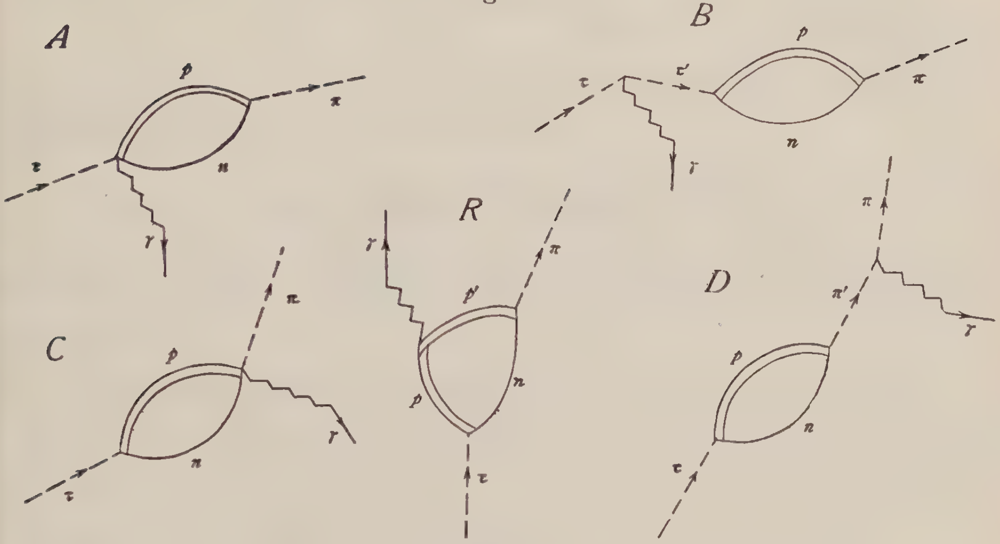
$$D=-\frac{-1}{(\hbar c)^2}\int_{-\infty}^{\sigma}d\omega'\int_{-\infty}^{\sigma'}d\omega''\,\{[[H_G(\tau,n),\ H_e'(\pi,\gamma)],\ H_g''(\pi,n)]$$
$$+[[H_G(\tau,n),\ H_g'(\pi,n)],\ H_e''(\pi,\gamma)]\}$$

$$R=-\frac{1}{(\hbar c)^2}\int_{-\infty}^{\sigma}d\omega'\int_{-\infty}^{\sigma'}d\omega''\,\{[[H_G(\tau,n),\ H_g'(\pi,n)],\ H_e''(\pi,\gamma)]$$
$$+[[H_G(\tau,n),\ H_e'(n,\gamma)],\ H_g''(\pi,n)]\}.$$

According to the Feynman's diagram, these terms correspond to the processes shown in Fig. 1.

P and Q are the processes where τ and π mesons interact with electromagnetic field respectively and moreover they contain the normal dependent terms. Owing to the Furry theorem there are many cases where these terms vanish. Their contributions are only for the following combinations of the coupling of τ and π mesons with nucleons

Fig. 1



$$(s, s),\ (ps, ps),\ (v, v),\ (pv, pv),\ (pv, ps),\ (v, t),\ (t, t)$$

where, for instance, t denotes the tensor coupling of vector (pseudovector) meson and pv means the pseudovector coupling and so on. To simplify the notations, we write the interactions $H_G(\tau, n)$ and $H_g(\pi, n)$ in the following form:

$$\begin{aligned} &H_G(\tau, n) = GUL^*,\ L^* = \bar{\psi}\, O_\tau \tau_{PN} \psi, \quad U = \text{wave function of } \tau \text{ meson}, \\ &H_g(\pi, n) = gV^* M,\ M = \bar{\psi}\, O_\pi \tau_{PN} \psi, \quad V = \text{wave function of } \pi \text{ meson}. \end{aligned} \tag{15}$$

Then term A becomes for various types of τ meson as follows,

(1) scalar (pseudoscalar) τ meson with vector (pseudovector) coupling

$$\begin{aligned} A = \frac{-G_2}{\varkappa_\tau} \frac{ge}{(\hbar c)^2} \int_{-\infty}^{\sigma} d\omega' \{ &<[P_\alpha{}^*, M']>_0 U A_\alpha V^{*\prime} + <[P_\beta{}^*, M']>_0 \\ &\times U A_\alpha N_\alpha N_\beta V^{*\prime} \}, \end{aligned} \tag{16}$$

(2) vector (pseudovector) τ meson with vector (pseudovector) coupling

$$A = -\frac{-G_1}{\varkappa_\tau{}^2} \frac{ge}{(\hbar c)^2} \int_{-\infty}^{\sigma} d\omega' <[P_\gamma{}^*, M']>_0 A_\alpha U_{\alpha\beta} N_\beta N_\gamma V^{*\prime}, \tag{17}$$

(3) vector (pseudovector) τ meson with tensor coupling,

$$\begin{aligned} A = -\frac{G_2}{2\varkappa_\tau} \frac{ge}{(\hbar c)^2} \int_{-\infty}^{\sigma} d\omega' \{ &<[S_{\alpha\beta}, M']>_0 (A_\alpha U_\beta - A_\beta U_\alpha) V'^* \\ &+ 2 <[S_{\alpha\gamma}, M']>_0 (A_\alpha U_\beta - A_\beta U_\alpha) N_\gamma N_\beta V^{*\prime}, \end{aligned} \tag{18}$$

$$P_\beta{}^* = i\bar{\psi}(\gamma_5)\gamma_\beta \tau_{PN} \psi, \quad S = -i\bar{\psi}(\gamma_5)\gamma_\alpha \gamma_\beta \tau_{NP} \psi.$$

$< \ >_0$ denotes the vacuum expectation value with respect to nucleon field in the intermediate state. For instance,

$$<[P_\beta^*,\ M']>_0=\frac{1}{2}\mathrm{Sp}\{S^{(1)}(x'-x)(\gamma_5)\gamma_\beta S(x-x')\,O_\pi$$
$$+S(x'-x)(\gamma_5)\gamma_\beta S^{(1)}(x-x')\,O_\pi\}. \tag{19}$$

In the same way, terms B are given for each case by

(1) scalar (pseudoscalar) τ meson with scalar (pseudoscalar) coupling

$$B=\frac{G_1ge}{(\hbar c)^2}\int_{-\infty}^{\sigma}d\omega'\int_{-\infty}^{\sigma}d\omega''<[W^*,M']>_0V^{*\prime}\left\{D_\tau(x-x'')\frac{\partial U''}{\partial x_\alpha''}-\frac{\partial D_\tau(x-x'')}{\partial x_\alpha''}U''\right\}A_\alpha'', \tag{20}$$

(2) scalar (pseudoscalar) τ meson with vector (pseudovector) coupling

$$B=\frac{-G_2ge}{\varkappa_\tau(\hbar c)^2}\int_{-\infty}^{\sigma}d\omega'\int_{-\infty}^{\sigma}d\omega''<[P_\beta^*,M']>_0V^{*\prime}\left\{\frac{\partial D_\tau(x-x'')}{\partial x_\beta''}\frac{\partial U''}{\partial x_\alpha''}A_\alpha''\right.$$
$$\left.-\frac{\partial^2D_\tau(x-x'')}{\partial x_\alpha''\partial x_\beta''}U''A_\alpha''\right\}, \tag{21}$$

(3) vector (pseudovector) τ meson with vector (pseudovector) coupling

$$-\frac{G_1ge}{(\hbar c)^2}\int_{-\infty}^{\sigma}d\omega'\int_{-\infty}^{\sigma'}d\omega''<[P_\gamma^*,M']>_0V^{*\prime}\left\{(\delta_{\beta\gamma}-\frac{1}{\varkappa_\tau^2}\,\frac{\partial^2}{\partial x_\beta''\partial x_\gamma''}D_\tau(x-x''))A_\alpha''U_{\alpha\beta}''\right.$$
$$\left.+\left(\delta_{\beta\gamma}\frac{\partial}{\partial x_\alpha}-\delta_{\beta\alpha}\frac{\partial}{\partial x_\gamma}\right)D_\tau(x-x'')A_\alpha''U''\right\}, \tag{22}$$

(4) vector (pseudovector) τ meson with tensor coupling

$$\frac{G_2ge}{\varkappa_\tau(\hbar c)^2}\int_{-\infty}^{\sigma}d\omega'\int_{-\infty}^{\sigma'}d\omega''<[S_{\alpha\gamma},M']>_0V^{*\prime}\left\{-A_\beta''\,\frac{\partial D_\tau(x-x'')}{\partial x_\gamma''}\chi_{\alpha\beta}''\right.$$
$$\left.+\frac{\partial^2D_\tau(x-x'')}{\partial x_\beta''\partial x_\gamma''}(A_\alpha''U_\beta''-A_\beta''U_\alpha'')\right\}. \tag{23}$$

Terms C and D are obtained by exchanging the roles of τ and π meson in A and B respectively. Term R is

$$R=-\frac{i}{2}\,\frac{Gge}{(\hbar c)^2}\int_{-\infty}^{\sigma}d\omega'\int_{-\infty}^{\sigma'}d\omega''\,\mathrm{Sp}\,\{S^{(1)}(x-x'')\,O_\pi\bar{S}(x''-x')\gamma_\alpha\bar{S}(x'-x)\,O_\tau$$
$$+\bar{S}(x-x'')\,O_\pi S^{(1)}(x''-x')\gamma_\alpha\bar{S}(x'-x)\,O_\tau$$
$$+\bar{S}(x-x'')\,O_\pi\bar{S}(x''-x')\gamma_\alpha S^{(1)}(x'-x)\,O_\tau\}UA_\alpha'V''. \tag{24}$$

The integrands of A and C contain the normal of the surface. The precise meaning of this integral can be given when we assign to each world point a space-like surface. Then the normal of the surface is uniquely defined at each world point. Using the relation

$$\int_{-\infty}^{\sigma}\partial_\alpha''\partial_\beta'' D(x-x'')\, F(x'')\, d\omega''=N_\alpha N_\beta F(x)+\int_{-\infty}^{\sigma} d\omega''\, D(x-x'')\, \partial_\alpha''\partial_\beta'' F(x''),$$

the normal dependent terms of A and C cancel one part of the term containing second derivative in B and D respectively. Then the total expressions are independent of the assignment of the surfaces.

3. Formal proof of gauge invariance

Though our proof may not be completely strict[8], it is possible to show that whole expressions are invariant under the transformation

$$A_\alpha\rightarrow A_\alpha+\partial_\alpha\Lambda, \qquad (\Box\Lambda=0). \tag{25}$$

(i) The cases when P and Q vanish.

Using the relation $(\gamma\partial+\varkappa)\, S^{(1)}(x-x')=0$ and $(\gamma\partial+\varkappa)\, \bar{S}(x-x')=-\delta(x-x')$ and integrating by part, the term containing Λ in R becomes

$$\begin{aligned} R;\quad -\frac{i}{2}\frac{Gge}{(\hbar c)^2}\int_{-\infty}^{\infty} d\omega'(\Lambda-\Lambda')\ \mathrm{Sp}\,\{&S^{(1)}(x'-x)\, O_\pi\, S(x-x')\, O_\tau \\ &+\bar{S}(x'-x)\, O_\pi\, S^{(1)}(x-x')\, O_\tau\}\, U\, V' \end{aligned} \tag{26}$$

which is proved to be always zero in this case. For example, the case $G_2(v)g_2(ps)\,e$(τ meson is vector meson with tensor coupling and π meson pseudoscalar meson with pseudovector coupling) becomes as follows;

$$\begin{aligned} \frac{-i}{4\varkappa_\tau\varkappa_\pi}\frac{G_2 g_2 e}{(\hbar c)^2}\int_{-\infty}^{\infty} d\omega'(\Lambda-\Lambda')\ \mathrm{Sp}\,\{&S^{(1)}(x-x')\gamma_5\gamma_\alpha\bar{S}(x'-x)\gamma_\beta\gamma_\tau \\ &+\bar{S}(x-x')\gamma_5\gamma_\alpha S^{(1)}(x'-x)\gamma_\beta\gamma_\tau\}\, U_{\beta\tau}\frac{\partial}{\partial x_\alpha}V^{*\prime}=0. \end{aligned}$$

(ii) The cases when P and Q do not vanish.

In these cases, adding to the terms (26) the contributions from P and Q, the whole expressions can also be shown to be invariant under gauge transformation. For instance, in the case $G_2(ps)\,g_1(pv)\,e$ each terms containing Λ are given by,

$$A;\quad \frac{G_2}{\varkappa_\tau}\frac{g_1 e}{(\hbar c)^2}\left(\frac{i}{2}\right)\int_{-\infty}^{\infty} d\omega'\, F(x-x')\, U\frac{\partial\Lambda}{\partial x_\alpha}V_\beta^{*\prime}$$

$$B;\quad \frac{G_2}{\varkappa_\tau}\frac{g_1 e}{(\hbar c)^2}\left(\frac{-i}{2}\right)\int_{-\infty}^{\infty} d\omega'\, F(x-x')\left(U\frac{\partial\Lambda}{\partial x_\alpha}+\frac{\partial U}{\partial x_\alpha}\Lambda\right)V_\beta^{*\prime}$$

$$D;\quad \frac{G_2}{\varkappa_\tau}\frac{g_1 e}{(\hbar c)^2}\left(\frac{i}{2}\right)\int_{-\infty}^{\infty} d\omega'\, F(x-x')\frac{\partial U}{\partial x_\alpha}\Lambda' V_\beta^{*\prime}$$

$$R;\quad \frac{G_2}{\varkappa_\tau}\frac{g_1 e}{(\hbar c)^2}\left(\frac{-i}{2}\right)\int_{-\infty}^{\infty} d\omega'\, F(x-x')\,(\Lambda'-\Lambda)\,\frac{\partial U}{\partial x_\alpha}V_\beta^{*\prime}$$

where

$$F(x-x')=\mathrm{Sp}\,\{S^{(1)}(x'-x)\gamma_5\gamma_\alpha S(x-x')\gamma_5\gamma_3+\bar{S}(x'-x)\gamma_5\gamma_\alpha S^{(1)}(x-x')\gamma_5\gamma_3\},$$

which cancel each other.

Summarizing the results of (i) and (ii), the gauge invariance seems to be formally secured for every case.

4. Formal proof of equivalence theorem

According to Case's general treatment of the equivalence theorem for photon-meson-nucleon coupling[9], equivalence is also expected because our case is of the order *Gge*.

To verify this, we put in R, $U=\partial_\alpha U$, $L=\gamma_5\gamma_\alpha$ and $G=(i/\chi_\tau)G_2$. U denotes the wave function of scalar (pseudoscalar) τ meson. For scalar (pseudoscalar τ meson with vector (pseudovector) coupling, the usual procedure gives for (24),

$$\begin{aligned}&R=O\ \text{(part equivalent to pseudoscalar coupling)}\\&+\frac{G_2}{\chi_\tau}\frac{g_1e}{(\hbar c)^2}\left(\frac{+1}{2}\right)\Big[UV^*\int_{-\infty}^{\infty}d\omega'\,A_\alpha'\,\mathrm{Sp}\,\{S^{(1)}(x-x')\gamma_\alpha S(x'-x)(\gamma_5)M\\&+\bar{S}(x-x')\gamma_\alpha S^{(1)}(x'-x)(\gamma_5)M\}+UA_\alpha\int_{-\infty}^{\infty}d\omega'\,V^{*\prime}\,\mathrm{Sp}\,\{S^{(1)}(x-x')\\&\times M\bar{S}(x'-x)\gamma_\alpha(\gamma_5)+\bar{S}(x-x')MS^{(1)}(x'-x)\gamma_\alpha(\gamma_5)\}\Big].\qquad(27)\end{aligned}$$

Using the regulator and removing the zeroth term in the expansion of the power $\Box^2/\chi^2$, the first term in the above bracket is zero by the relation $\Box^2 A_\mu=0$. The second term, as is easily seen, cancels the first term in (16) exactly. For the remaining terms B and D identities are easily verified. Thus, formally identity holds if we admit the regulator and drop the undesirable terms. It should be, however, remembered that in these proof expressions which may contain divergence are removed by the law of conservation of energy and momentum or usual canonical transformation and so our procedure cannot be said completely rigorous.

5. Calculations and the use of Regulating method

Using the representations of the Δ and $\Delta^{(1)}$ function proposed by Schwinger[2] and resolving the Hamiltonian density for the order *Gge* into gauge and non-gauge parts, it is shown that non gauge part vanishes except the leading term in the expansion of $(\Box^2/\chi^2)$. For the typical cases we get the following expressions.

(i) Case of $G_2(ps)\,g_1(pv)\,e$

$(P+Q)=(P+Q)_1+(P+Q)_2,$

$$(P+Q)_1=\left(\frac{G_2}{\kappa_\tau}\right)\frac{g_1 e}{(\hbar c)^2}\frac{(-i)}{(2\pi)^2}\frac{1}{2}\kappa^2\left[+\left\{2\int_0^\infty\frac{\sin w}{w^2}dw+2\int_0^\infty\frac{\cos w}{w}dw+\frac{1}{3}\frac{\kappa_\tau^2}{\kappa^2}\right\}\right.$$

$$\left.\times\frac{1}{\kappa_\pi^2}UF_{\alpha\beta}\frac{\partial V_\beta{}^*}{\partial x_\alpha}+\left\{\frac{2}{3}\frac{\kappa_\pi^2}{\kappa^2}\int_0^\infty\frac{\cos w}{w}dw+\frac{1}{3}(\kappa_\pi^2+\kappa_\tau^2)\right\}UA_\alpha U_\alpha{}^*\right],$$

$$(P+Q)_2=\frac{G_2}{\kappa_\tau}\frac{g_1 e}{(\hbar c)^2}\frac{i}{(2\pi)^2}\frac{1}{8}\frac{\kappa_\tau^2}{\kappa_\pi^2}\frac{\kappa_\tau^4}{\kappa^4}\int_{-1}^{1}\frac{y^2(y^2-1)}{1-\frac{1-y^2}{4}\frac{\kappa_\tau^2}{\kappa^2}}dyUF_{\alpha\beta}\frac{\partial V_\beta{}^*}{\partial x_\alpha},$$

$$R=R_1+R_2,$$

$$R_1=\frac{G_2}{\kappa_\tau}\frac{g_1 e}{(\hbar c)^2}\frac{(-i)}{(2\pi)^2}\frac{1}{3}\left[\kappa_\pi^2\int_0^\infty\frac{\cos w}{w}dw\,UA_\alpha V_\alpha{}^*-\frac{7}{6}\kappa_\pi^2UA_\alpha V_\alpha{}^*\right.$$

$$\left.-UF_{\alpha\beta}\frac{\partial V_\beta{}^*}{\partial x_\alpha}-\kappa_\tau^2UA_\alpha V_\alpha{}^*\right],$$

$$R_2=\frac{G_2}{\kappa_\tau}\frac{g_1 e}{(\hbar c)^2}\frac{(-i)}{(2\pi)^2}\frac{1}{2^4}\int_1^\infty\frac{du}{u^6}\int_{-1}^{1}dv\left\{1+\frac{(1-v)(1-u)}{2u^2}\frac{\kappa_\pi^2}{\kappa^2}-\frac{(1-v^2)}{4u^2}\frac{\kappa_\tau^2}{\kappa^2}\right\}^{-1}$$

$$\times\left[\frac{4}{3}(1-v)(-3+6u+5v-4uv-3u^2)\frac{\kappa_\pi^2}{\kappa^2}\right.$$

$$\left.+(1-v^2)\left(4-4u-\frac{16}{3}u\right)\frac{\kappa_\tau^2}{\kappa^2}\right]UF_{\alpha\beta}\frac{\partial V_\beta{}^*}{\partial x_\alpha}.$$

(ii) Case of $G_1(ps)g_1(pv)e$

$$(P'+Q')=(P'+Q')_1+(P'+Q')_2+(P'+Q')_3,$$

$$(P'+Q')_1=\frac{G_1 g_1 e}{(\hbar c)^2}\frac{(-i)}{(2\pi)^2}\kappa\int_0^\infty\frac{\cos w}{w}dwU\left(A_\alpha V_\alpha{}^*+\frac{1}{\kappa_\pi^2}F_{\alpha\beta}\frac{\partial V_\beta{}^*}{\partial x_\alpha}\right),$$

$$(P'+Q')_2=\frac{G_1 g_1 e}{(\hbar c)^2}\frac{(-i)}{(2\pi)^2}\frac{1}{6}\kappa\frac{\kappa_\tau^2}{\kappa^2}\frac{1}{\kappa_\pi^2}F_{\alpha\beta}\frac{\partial V_\beta{}^*}{\partial x_\alpha},$$

$$(P'+Q')_3=\frac{G_1 g_1 e}{(\hbar c)^2}\frac{i}{(2\pi)^2}\frac{1}{16}\frac{\kappa}{\kappa_\pi^2}\frac{\kappa_\tau^4}{\kappa^4}\int_{-1}^{1}\frac{y^2(y^2-1)}{1-\frac{1-y^2}{4}\frac{\kappa_\tau^2}{\kappa^2}}dy\,UF_{\alpha\beta}\frac{\partial V_\beta{}^*}{\partial x_\alpha},$$

$$R'=R_1'+R_2'+R_3',$$

$$R_1'=\frac{G_1 g_1 e}{(\hbar c)^2}\frac{i}{(2\pi)^2}\kappa\left[\int_0^\infty\frac{\cos w}{w}dw-\frac{1}{2}\right]UA_\alpha U_\alpha{}^*,$$

$$R_2'=\frac{G_1 g_1 e}{(\hbar c)^2}\frac{i}{(2\pi)^2}\frac{1}{6}\frac{1}{\kappa}UF_{\alpha\beta}\frac{\partial V_\beta{}^*}{\partial x_\alpha},$$

$$R_3'=\frac{G_1 g_1 e}{(\hbar c)^2}\frac{-i}{(2\pi)^2}\frac{1}{2^3}\frac{1}{\kappa}\int_1^\infty\frac{du}{u^6}\times$$

$$\times\int_{-1}^{1} dv \frac{(1-v-u)\left\{-2(1-v)(1-u)\frac{x_\tau^2}{x^2}+(1-v^2)\frac{x_\tau^2}{x^2}\right\}}{1+\frac{(1-v)(1-u)}{2u^2}\frac{x_\pi^2}{x^2}-\frac{(1-v^2)}{4u^2}\frac{x_\tau^2}{x^2}} UF_{\alpha 3}\frac{\partial V_\beta^*}{\partial x_\alpha}.$$

Those terms which are divergent and gauge invariant can be removed by the following conditions imposed on regulator which are written in the bracket.

$$(P+Q)_1,\ \left(\int\rho(x)\,x\log|x|\,dx=0,\ \int\rho(x)\log|x|\,dx=0,\ \int\rho(x)\,dx=0\right),$$

$$(P'+Q')_1,\ \left(\int\sqrt{x}\,\rho(x)\,dx=0\right),$$

$$R_1,\ \left(\int\rho(x)\,dx=0,\ \int\rho(x)\log|x|\,dx=0\right),\ R_1',\ \left(\int\sqrt{x}\,\rho(x)\,dx=0\right).$$

To secure the equivalence theorem we must remove the terms $(P'+Q')_2$ and R_2' which are gauge invariant and seem to be unconditionally convergent. In our paper I[10], we adopted the procedure in the use of regulator which lets the identity hold.

But, as was stated before, our formal proof of identity being not completely strict, we cannot decide whether we should remove these terms or abandon the equivalence theorem.

(iii) Case of $G_2(v)g_2(ps)e$

This case was treated by Finkelstein[11] in the model of Schwinger's mixture theory. Contrary to his results, there are no contributions from P and Q and moreover no logarithmic divergence in our calculation.

$$R=R_1+R_2,$$

$$R_1=\frac{G_2g_2e}{(\hbar c)^2}\frac{(i)}{x_\tau x_\pi}\frac{1}{(2\pi)^2}\frac{1}{2^5}\,x\,\mathrm{Sp}\,(\gamma_5\gamma_\alpha\gamma_3\gamma_\tau\gamma_\mu)\,U_{3\tau}A_\mu\frac{\partial}{\partial x_\alpha}V^*,$$

$$R_2=\frac{G_2g_2e}{(\hbar c)^2}\frac{(-i)}{x_\tau x_\pi}\frac{1}{(2\pi)^2}\frac{1}{2^5}\frac{1}{x}\int_1^\infty\frac{du}{u^5}\int_{-1}^{1}dv\,[(1+v)(1+v-u)$$

$$\times\mathrm{Sp}\,(\gamma_5\gamma_\mu\gamma_3\gamma_\tau\gamma_\rho)\,U_{3\tau}\frac{\partial}{\partial x_\alpha}F_{\rho\mu}\partial_\alpha V^*$$

$$-(u^2+v^2-1)\,x_\pi^2\,\mathrm{Sp}\,(\gamma_5\gamma_\mu\gamma_3\gamma_\tau\gamma_\rho)\,U_3F_{\rho\mu}V^*]$$

$$\times\left\{1+\frac{(1-v)(1-u)}{2u^2}\left(\frac{x_\pi}{x}\right)^2-\frac{(1-v^2)}{4u^2}\frac{x_\tau^2}{x^2}\right\}^{-1}$$

Term R_1 is removed by the condition $\int\sqrt{x}\,\rho(x)\,dx=0$.

In every case, it turns out that the conditions $\int\rho(x)\,dx=0$,

$$\int\sqrt{x}\,\rho(x)\,dx=0,\quad\int\rho(x)\log|x|\,dx=0\ \text{ and }\ \int\rho(x)\,x\log|x|\,dx=0$$

are sufficient enough to make all results convergent and gauge invariant and moreover the contributions of the vector coupling of scalar meson vanish. But as was shown in our paper I and II, under these conditions of regulator there are some cases where the equivalence of the two interaction forms of pseudoscalar meson does not hold.

6. Selection rules and the life-times of these processes

As stated in the introduction, there exist no such selection rules as found in the case of symmetrical interactions. After calculation, it turns out that the following combinations of τ and π meson are forbidden.

(i) The cases when both τ and π mesons have spin 0.

So that equivalence theorem holds in this forbidden case.

(ii) The cases when τ or π meson is the scalar meson with vector coupling. In these cases, the use of regulator is made and the diverging and non-gauge invariant terms are removed.

The rule (i) is easily understood. Under the request of gauge invariancy the interaction Hamiltonians in these cases must be the linear combinations of $F_{\alpha\beta}\partial_\alpha U\partial_\beta\varphi^*$, $F_{\alpha\beta}\partial_\alpha\partial_\beta U\varphi^*$ and $F_{\alpha\beta}U\partial_\alpha\partial_\beta\varphi^*$ which are identically zero.

In the following, we show the decay life-times of these processes for the typical cases after the removal of the diverging and non-gauge invariant terms.

We made the following assumptions

$$\frac{G^2}{\hbar c}\sim 10^{-3},\quad \frac{g^2}{\hbar c}\sim 10^{-1},\quad x_\tau=900\ m_e,\quad x_\pi=300\ m_e.$$

$G_1(v)\ g_1(v)\ e$	4.2×10^{-14} sec	$G_2(v)\ g_2(ps)\ e$	2.6×10^{-12} sec
$G_1(pv)\ g_2(pv)\ e$	2.3×10^{-13} sec	$G_1(v)\ g_1(ps)\ e$	2.8×10^{-14} sec
$G_1(ps)\ g_1(pv)\ e$	1.5×10^{-15} sec	$G_1(s)\ g_1(v)\ e$	4.0×10^{-15} sec
$G_2(ps)\ g_2(pv)\ e$	1.3×10^{-13} sec	$G_1(s)\ g_2(v)\ e$	3.1×10^{-13} sec

Using the regulator, the life times seem to be shorter than 10^{-11} sec. in every case under the above assumptions.

7. Conclusion

According to the above analysis, the γ decay of a charged τ meson seems to be too short to be detected in experiments. In order to explain the experiments by Powell and others who reported the process $\tau^\pm\rightarrow\pi^\pm+\pi^++\pi^-$, we must search for the combinations of τ and π mesons where the processes discussed in II and here are forbidden. These are discussed in detail in the introduction. Using these analysis of the third order decay processes of a τ meson, we shall proceed to the case $\tau^\pm\rightarrow\pi^\pm+\pi^++\pi^-$ which will be discussed in our paper IV.

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